On integrability of the sub-Riemannian geodesic flow for Goursat distribution.

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Formulation of the problem

Consider the following optimal control problem:

$$\dot{q} = u_1 f_1(q) + u_2 f_2(q),$$

where $q = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ determines the state of the system,

$$f_1 = (1, 0, -x_2, -x_3, ..., -x_{n-1})^T,$$

$$f_2 = (0, 1, 0, 0, ..., 0)^T,$$

$u = (u_1, u_2) \in \mathbb{R}^2$ is a control, boundary conditions:

$$q(0) = q_0,\ q(t_1) = q_1,$$

quality functional:

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to min.$$

Goursat distribution

Vector fields f_1, f_2 generate n-dimensional nilpotent Lie algebra called the Goursat algebra.

$$f_3 = \frac{\partial f_2}{\partial q} f_1 - \frac{\partial f_1}{\partial q} f_2 = [f_1, f_2] = (0, 0, 1, 0, ..., 0)^T,$$

$$f_4 = \frac{\partial f_3}{\partial q} f_1 - \frac{\partial f_1}{\partial q} f_3 = [f_1, f_3] = (0, 0, 0, 1, ..., 0)^T,$$

$$...$$

$$f_n = \frac{\partial f_{n-1}}{\partial q} f_1 - \frac{\partial f_1}{\partial q} f_{n-1} = [f_1, f_{n-1}] = (0, 0, ..., 0, 1)^T.$$

$f_1, f_2, ..., f_n$ are basis left-invariant fields on the Goursat group. The initial system turned out to be completely controllable (it follows from the Rashevskii – Chow theorem). Existence of optimal solutions follows from the Filippov theorem.

Hamiltonian equations

$$\begin{cases} \dot{x_1} = u_1, \\ \dot{x_2} = u_2, \\ \dot{x_3} = -u_1 x_2, \\ \dots \\ \dot{x_n} = -u_1 x_{n-1}. \end{cases}$$

Introduce the conjugate momenta $(p_1, ..., p_n)$ and construct the hamiltonian:

$$H(q, p, u) = u_1 \langle p, f_1 \rangle + u_2 \langle p, f_2 \rangle - \frac{u_1^2 + u_2^2}{2}.$$

It follows from the Pontryagin maximum principle that $H_{u_1} = H_{u_2} = 0$, i.e.

$$u_1 = \langle p, f_1 \rangle, u_2 = \langle p, f_2 \rangle.$$

Thus H takes the form:

$$H(q, p, u) = \frac{u_1^2 + u_2^2}{2} = \frac{\dot{x_1}^2 + \dot{x_2}^2}{2} = \frac{\langle p, f_1 \rangle^2 + \langle p, f_2 \rangle^2}{2}.$$

Hamiltonian equations

The following equations define dynamics of the initial system:

$$\begin{cases} \dot{x_1} = p_1 - x_2 p_3 - ... - x_{n-1} p_n, \\ \dot{x_2} = p_2, \\ \dot{x_3} = -x_2 \dot{x_1}, \\ ... \\ \dot{x_n} = -x_{n-1} \dot{x_1}. \end{cases} \tag{1}$$

$$\begin{cases} \dot{p_1} = 0, \\ \dot{p_2} = p_3 \dot{x_1}, \\ ... \\ \dot{p_{n-1}} = p_n \dot{x_1}, \\ \dot{p_n} = 0. \end{cases} \tag{2}$$

Theorem

(1), (2) is the completely integrable system (in the sense of Liouville).

Integrability

Introduce $s(t)$ such that $\dot{s} = u_1$, then $s(t) = x_1(t)$. Rewrite (2) in terms of s:

$$\begin{cases} \dot{p_1} = 0, \\ \dot{p_2} = p_3, \\ \dots \\ \dot{p_{n-1}} = p_n, \\ \dot{p_n} = 0. \end{cases}$$

The first integrals of (1), (2) are of the form:

$$\begin{cases} F_n = p_n, \\ F_{n-1} = p_{n-1} - F_n x_1, \\ F_{n-2} = p_{n-2} - F_{n-1} x_1 - F_n \frac{x_1^2}{2!}, \\ \dots \\ F_2 = p_2 - F_3 x_1 - F_4 \frac{x_1^2}{2!} - \dots - F_n \frac{x_1^{n-2}}{(n-2)!}, \\ H = \frac{1}{2}(p_2^2 + (p_1 - x_2 p_3 - \dots - x_{n-1} p_n)^2). \end{cases} \quad (3)$$

All of them are functionally independent and are in pairs in involution.

Integrability

Theorem (Arnold – Liouville)
Let M be a symplectic manifold, dim $M = 2n$. *Assume that smooth functions* $F_1, ..., F_n : M \to \mathbb{R}$ *are functionally independent and are in pairs in involution:* $\{F_i, F_j\} = 0 \ (1 \le i, j \le n)$. *Then*
1)every connected component M_α *is diffeomorphic to* $\mathbb{R}^k \times \mathbb{T}^{n-k}$,
2)there exist such coordinates $(y_1, ..., y_k; \varphi_1, ..., \varphi_{n-k} \ mod \ 2\pi)$ *on* $\mathbb{R}^k \times \mathbb{T}^{n-k}$ *that the Hamiltonian equations (1) take the form:*

$$\dot{y_m} = c_m, \dot{\varphi}_s = \varpi_s \ (c_m, \varpi_s = const).$$

Introduce the following change of coordinates:

$$\begin{cases} P_1 = p_1 - x_2 p_3 - x_3 p_4 - \ldots - x_{n-1} p_n, \\ P_n = p_n, \\ P_{n-1} = p_{n-1} - P_n x_1, \\ P_{n-2} = p_{n-2} - P_{n-1} x_1 - P_n \frac{x_1^2}{2!}, \\ \ldots \\ P_3 = p_3 - P_4 x_1 - P_5 \frac{x_1^2}{2!} - \ldots - P_n \frac{x_1^{n-3}}{(n-3)!}, \\ P_2 = p_2. \end{cases}$$

Then (1), (2) take the form:

$$\begin{cases} \dot{x_1} = P_1, \\ \dot{x_2} = P_2, \\ \dot{x_3} = -P_1 x_2, \\ \ldots \\ \dot{x_n} = -P_1 x_{n-1}, \end{cases} \tag{4}$$

$$\begin{cases} \dot{P_1} = -P_2 (P_3 + P_4 x_1 + \ldots + P_n \frac{x_1^{n-3}}{(n-3)!}), \\ \dot{P_2} = P_1 (P_3 + P_4 x_1 + \ldots + P_n \frac{x_1^{n-3}}{(n-3)!}), \\ \dot{P_3} = 0, \\ \ldots \\ \dot{P_{n-1}} = 0, \\ \dot{P_n} = 0. \end{cases} \tag{5}$$

Level surfaces

Integrals have the form:

$$\begin{cases} F_n = P_n, \\ F_{n-1} = P_{n-1}, \\ \dots \\ F_3 = P_3, \\ F_2 = P_2 - P_3 x_1 - \dots - P_n \frac{x_1^{n-2}}{(n-2)!}, \\ F_1 = H = \frac{1}{2}(P_1^2 + P_2^2). \end{cases} \tag{6}$$

Fix some values of the integrals: $H = \frac{C_1^2}{2},\ F_j = C_j,\ j = 2,\ \dots,n.$

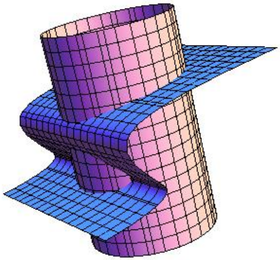
$(x_2,\ \dots,x_n) \in \mathbb{R}^{n-1}$.
$(P_3,\ \dots,P_n)$ are fixed as the first integrals.

2 equations on (x_1, P_1, P_2) left:

$$\begin{cases} P_2 = C_2 + P_3 x_1 + \dots + P_n \frac{x_1^{n-2}}{(n-2)!}, \\ C_1^2 = (P_1^2 + P_2^2). \end{cases} \tag{7}$$

Level surfaces

$$\begin{cases} P_2 = C_2 + C_3 x_1 + \ldots + C_n \frac{x_1^{n-2}}{(n-2)!}, \\ C_1^2 = (P_1^2 + P_2^2). \end{cases}$$



There are some closed curves lying in the intersection of this two surfaces, the number of them being not greater than $(n-2)$. They are proved to be $\mathbb{S}^1$. The level surfaces in regular points are $\mathbb{S}^1 \times \mathbb{R}^{n-1}$.

"Moment map"and its critical points

$$\Phi : (x, P) \mapsto \begin{pmatrix} F_n \\ ... \\ F_1 \end{pmatrix}$$

Jacoby matrix $J_{n\times 2n} = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial P})$ is of the form:

$$\begin{pmatrix} 0 & ... & 0 & 0 & ... & 0 & 0 & 0 & 1 \\ 0 & ... & 0 & 0 & ... & 0 & 0 & 1 & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & ... & 0 & 0 & 0 & 1 & 0 & ... & 0 \\ w_0 & ... & 0 & 0 & 1 & w_1 & w_2 & ... & w_{n-2} \\ 0 & ... & 0 & P_1 & P_2 & 0 & 0 & ... & 0 \end{pmatrix}$$

Here

$$w_0 = -(P_3 + P_4 x_1 + ... + P_n \frac{x_1^{n-3}}{(n-3)!}), \;\; w_k = -\frac{x_1^k}{k!}, \;\; k > 0.$$

"Moment map"and its critical points

$P_1 \neq 0 \;\Rightarrow$ rank $J = n$.

$P_1 = P_2 = 0 \;\Rightarrow\; H = 0$.

Let $P_1 = 0,\; P_2 \neq 0$, i.e. $H = \frac{1}{2}P_2^2 > 0$. Consider the following minor (two last lines are transposed):

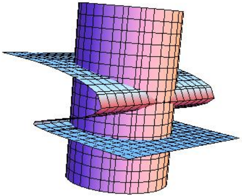
$$\begin{pmatrix} 0 & ... & 0 & 0 & 0 & 1 \\ 0 & ... & 0 & 0 & 1 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & P_2 & 0 & ... & 0 & 0 \\ w_0 & 1 & w_1 & w_2 & ... & w_{n-2} \end{pmatrix}$$

Thus all the critical points can be described by the following system of equations:

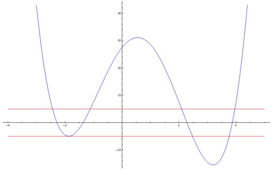
$$\begin{cases} P_1 = 0, \\ w_0 = \frac{\partial F_2}{\partial x_1} = 0. \end{cases} \qquad (8)$$

Critical points and degenerate level surfaces

Geometrically it means that circles have common points.



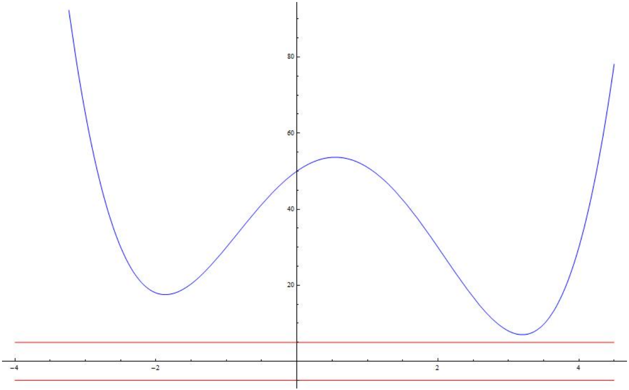
This is how it looks like on the plane (x_1, P_2).



The level surfaces in critical points are $\mathbb{R}^{n-1}$.

Critical points and degenerate level surfaces

The following case may also take place:



Corollary

What we have done:

Integrability of the sub-Riemannian geodesic flow has been proved.
The first integrals have been found explicitly.
The level surfaces have been constructed.
The critical points of "moment map"have been found.

What we need:

Compactification of the level surfaces.
Search for the "action – angle"coordinates and integration of the differential equations of motion.

Thank you for attention!