Geometric and analytical properties of Carnot–Carathéodory spaces under minimal smoothness

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Carnot manifolds

Equiregular Carnot–Carathéodory space:
N-dimensional Riemannian manifold $\mathcal{M}$ with a sequence of C^1-smooth distributions

$$\mathcal{D} = \mathcal{D}_1 \subsetneq \mathcal{D}_2 \subsetneq \cdots \subsetneq \mathcal{D}_r = T\mathcal{M}$$

such that $[\mathcal{D}_i, \mathcal{D}_j](q) \subset \mathcal{D}_{i+j}(q)$ for every $q \in \mathcal{M}$, $i + j \leq r$.

Carnot manifold:
If also

$$\mathcal{D}_k = \mathrm{span}\{\mathcal{D}_{k-1}, [\mathcal{D}_i, \mathcal{D}_j] : i + j = k\}$$

for $k = 2, \ldots, r$.

In terms of vector fields

Locally, we can choose a C^1-smooth basis $X_1, \dots, X_N$ in $T\mathcal{M}$ such that $\mathcal{D}_k = \mathrm{span}\{X_1, \dots, X_{\dim \mathcal{D}_k}\}$.

Assign $d(k) = \min\{m : X_k \in \mathcal{D}_m\}$.

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Carnot–Carathéodory space:

$$[X_i, X_j](x) = \sum_{k:\, d(k) \leq d(i)+d(j)} c_{ij}^k(x) X_k(x), \quad c_{ij}^k \in C^0.$$

Carnot manifold:

$$X_k = \sum_{m:\, d(m)<d(k)} a_m X_m + \sum_{i,j:\, d(i)+d(j)=d(k)} b_{ij}[X_i, X_j], \quad a_m, b_{ij} \in C^0,$$

for $k > \dim \mathcal{D}$.

Examples

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An example of 3-step structure in $\mathbb{R}^4$.

$$\text{Take } f(x,y,z,w) \in C^1(\mathbb{R}^4), \quad \frac{\partial f}{\partial x} \neq 0,$$
$$\text{and } g(y,z,w) \in C^1(\mathbb{R}^3), \quad \frac{\partial g}{\partial y} \neq 0.$$

$$\begin{aligned} X &= \partial_x, & \mathcal{D} &= \text{span}\{X, Y\}, \\ Y &= \partial_y + f\partial_z + fg\partial_w, & [X, Y] &= \frac{\partial f}{\partial x} \cdot Z, \\ Z &= \partial_z + g\partial_w, & [Y, Z] &= \frac{\partial g}{\partial y} \cdot W - Zf \cdot Z, \\ W &= \partial_w. \end{aligned}$$

"Box" quasimetric

The mapping

$$\theta_g : (t_1, \ldots, t_n) \mapsto \exp(t_1 X_1 + \cdots + t_N X_N)(g)$$

is a C^1-diffeomorphism of $U(0)$ onto $V(g)$.

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If $y = \theta_x(t_1, \ldots, t_N)$ define

$$d_\infty(x, y) = \max\{|t_1|^{\frac{1}{d(1)}}, \ldots, |t_N|^{\frac{1}{d(N)}}\}.$$

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Theorem (Karmanova, Vodopyanov)

Locally d_∞ is a quasimetric:

$$d_\infty(x, y) \le Q(d_\infty(x, z) + d_\infty(z, y))$$

for some $Q \ge 1$.

Proving local connectivity

Fix $g \in \mathcal{M}$. The mapping

$$\widehat{\theta}_g(t_1, \ldots, t_n) = \exp(t_N \widehat{X}_N^g) \circ \cdots \circ \exp(t_1 \widehat{X}_1^g)(g)$$

is a C^1-diffeomorphism $U(0) \subset \mathbb{R}^N$ onto $V(g) \subset \mathcal{G}^g \cap \mathcal{M}$.

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By Rashevsky–Chow theorem

$$\exp(\widehat{X}_k^g)(g) = \exp(a_{L,k} \widehat{X}_{L,k}^g) \circ \dots \circ \exp(a_{L,1} \widehat{X}_{L,1}^g)(g)$$

where all $\widehat{X}_{L,i}^g \in \{\widehat{X}_1^g, \dots, \widehat{X}_{\dim \mathcal{D}}^g\}$.

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We have an estimate

$$d_\infty(\widehat{\Phi}_g(t_1, \ldots, t_N), \Phi_g(t_1, \ldots, t_N)) = o(\|(t_1, \ldots, t_N)\|)$$

Theorem on connectivity

Theorem (B., Vodopyanov, 2012)

Fix $g \in \mathcal{M}$. Let $X_1, \dots, X_{\dim \mathcal{D}}$ be a basis in $\mathcal{D}$. There exists a neighborhood $U(g)$ such that for every pair of points x, $y \in U(g)$

$$y = \exp(a_L X_{j_L}) \circ \cdots \circ \exp(a_2 X_{j_2}) \circ \exp(a_1 X_{j_1})(x)$$

where $1 \leq j_k \leq \dim \mathcal{D}$, $k = 1, \dots, L$, $L \in \mathbb{N}$ and $|a_i| \leq C\, d_\infty(x, y)$. Here constants L and C do not depend on x, y.

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We have an estimate

$$c\, d_\infty(x, y) \overset{?}{\leq} d_{cc}(x, y) \leq LC\, d_\infty(x, y)$$

Ball–Box theorem and corollaries

Theorem (Karmanova, Vodopyanov)

Locally, there are $0 < C_1 \leq C_2 < \infty$ *such that*

$$Box(x, C_1 r) \subset B(x, r) \subset Box(x, C_2 r).$$

Corollaries:

- The Haussdorff dimension of $\mathcal{M}$ is

$$\nu = \sum_{i=1}^{N} d(i) = \sum_{k=1}^{r} k(\dim \mathcal{D}_k - \dim \mathcal{D}_{k-1}), \quad (\mathcal{D}_0 = \{0\}).$$

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- The (spherical) Haussdorff measure $\mathcal{H}^\nu$ is locally doubling

$$\mathcal{H}^\nu(B(x,2r)) \leq C\, B(x,r).$$

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$$C_3 \left(\frac{r}{r_0}\right)^\nu \le \frac{\mathcal{H}^\nu(B(x,r))}{\mathcal{H}^\nu(B(x,r_0))} \le C_4 \left(\frac{r}{r_0}\right)^\nu, \quad 0 < r \le r_0.$$

Sobolev spaces in metric spaces

(X, d, μ) — metric measure space.

Define $L_p(X)$, $1 \le p \le \infty$, as a set of μ-measurable functions f such that

$$\|f\|_{L_p(X)} = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} < \infty.$$

Let $f \in L_p(X)$. Upper gradient is a function $0 \le g \in L_p(X)$ such that

$$|f(x) - f(y)| \le d(x, y)(g(x) + g(y)) \quad \text{a. e.}$$

Define $W_p^1(X)$ as a space of $f \in L_p(X)$ such that

$$\|f\|_{W_p^1(X)} \le \|f\|_{L_p(X)} + \inf_g \|g\|_{L_p(X)} < \infty.$$

When does it behave like "normal" Sobolev space?

P. Hajłasz, 2000

Many results of the classical theory still hold in (X, d, μ) if

1. There are $s > 0$ and $C_\mu > 0$ such that

$$\frac{\mu(B(x,r))}{\mu(B(x,r_0))} \geq C_\mu \left(\frac{r}{r_0}\right)^s, \quad r \leq r_0.$$

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2. There are $q > 0$, $\sigma \geq 1$, $C_p > 0$ such that

$$\int_{B(x,r)} |f - f_B| \, d\mu \leq C_p r \Big(\int_{B(x,\sigma r)} g^q \, d\mu \Big)^{\frac{1}{q}}.$$

Our case

The metric measure space is $(\mathcal{M}, d_{cc}, \mathcal{H}^\nu)$.

The role of gradient is played by horizontal gradient

$$\nabla_H f = (X_1 f, \ldots, X_{\dim \mathcal{D}} f).$$

An open bounded set $\Omega \subset \mathcal{M}$ is called John domain of class $J(a, b)$, $0 < a \leq b$, if there is $x_0 \in \Omega$ such that every $x \in \Omega$ can be connected by a curve $\gamma : [0, L] \to \Omega$, parametrised by arc length, such that $\gamma(0) = x$, $\gamma(L) = x_0$, $L \leq b$ and

$$\operatorname{dist}(\gamma(s), \partial\Omega) \geq \frac{as}{L} \quad \text{for all } s \in [0, L].$$

Poincaré inequality

Additional regularity: partial derivatives of vector fields X_i have common modulus of continuity $\omega(t)$ such that $\int\limits_0^r \frac{\omega(t)}{t}\,dt < \infty$.
(e. g. $X_i \in C^{1,\alpha}$, $0 < \alpha \leq 1$)

Theorem (B., 2013)

Let $g \in \mathcal{M}$ and $1 \leq p < \infty$. Then there are $C_p > 0$ and $r_0 > 0$ such that for every John domain $\Omega \subset B(g, r_0)$ of class $J(a, b)$, $0 < a \leq b$ and for each $f \in C^\infty(\Omega)$

$$\|f - f_\Omega\|_{L_p(\Omega)} \leq C_p \left(\tfrac{b}{a}\right)^\nu \mathrm{diam}(\Omega) \|\nabla_H f\|_{L_p(\Omega)}$$

where $f_\Omega = \frac{1}{\mathcal{H}^\nu(\Omega)} \int_\Omega f\, d\mathcal{H}^\nu$.

Corollaries from Poincaré inequality

Let f, $\nabla_H f \in L_p(\Omega)$ where $1 \le p < \infty$.
(1) If $p < \nu$ then for $q = \frac{\nu p}{\nu - p}$ we have

$$\|f - f_\Omega\|_{L_q(\Omega)} \le C_1 (\tfrac{b}{a})^\nu \|\nabla_H f\|_{L_p(\Omega)};$$

(2) If $p = \nu > 1$ then

$$\fint_\Omega \exp\Big\{\Big(\frac{C_2 \mathcal{H}^\nu(\Omega)^{\frac{1}{\nu}} |f - f_\Omega|}{(\frac{b}{a})^\nu \|\nabla_H f\|_{L_\nu(\Omega)}}\Big)^{\frac{\nu}{\nu-1}}\Big\}\, d\mathcal{H}^\nu \le C_3;$$

(3) If $p > \nu$ then f is locally Hölder continuous

$$|f(x) - f(y)| \le C_4 (\tfrac{b}{a})^{\nu + \frac{\nu}{p}} d_{cc}(x,y)^{1 - \frac{\nu}{p}} \|\nabla_H f\|_{L_p(\Omega)}.$$

Here C_1, C_2, C_3, C_4 depend only on p and $\mathcal{M}$.