Optimal R&D Policy in a Model Based on Exhaustible Resources

Konstantin Besov

Steklov Mathematical Institute

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Introduction

- We consider an infinite-horizon economic growth problem of optimal dynamic allocation of economic resources between the sectors in a model of a two-sector economy.
- Although rather simplified, the model seems to be of economic interest, and it also illustrates the application of quite recent results in the abstract optimal control theory.
- The model is a development of a simpler (deterministic) model proposed and studied earlier jointly with Sergey Aseev and Serguei Kaniovski:

  S. Aseev, K. Besov, S. Kaniovski, in *Green Growth and Sustainable Development* (Springer, Berlin, 2013), Dynamic Modeling and Econometrics in Economics and Finance **14**, pp. 3–30.

  The extended ("randomized") model that will be presented below is also a result of discussions with S. Aseev and S. Kaniovski.

Deterministic model

1. The population (labor resource) $L = \mathrm{const}$ is fixed.
2. At each instant $t \geq 0$ the economy produces output $Y(t) > 0$ which is assumed to be described by a Cobb–Douglas production function:
$$Y(t) = A(t)^{\varkappa}[L - L^A(t)]^{\alpha} R_1(t)^{1-\alpha}, \qquad \alpha \in (0,1), \quad \varkappa > 0.$$
3. The amount of new knowledge (technologies) produced at time t depends on the hitherto accumulated knowledge $A(t)$, the number of researchers $L^A(t)$ and the portion $R_2(t)$ of the exhaustible resource used in research:
$$\dot{A}(t) = A(t)^{\theta}[L^A(t)]^{\eta} R_2(t)^{1-\eta}, \qquad \eta \in (0,1], \quad \theta \in (0,1].$$
4. The stock $S_0 > 0$ of the exhaustible resource is finite. This implies the following integral constraint on control variables:
$$\int_0^{\infty} \big[R_1(t) + R_2(t)\big]\, dt \leq S_0.$$
5. We take a discounted logarithmic utility function of the output as a measure of *welfare*:
$$J(Y(\cdot)) = \int_0^{\infty} e^{-\rho t}\big\{\ln[Y(t)]\big\}\, dt \quad \text{where } \rho > 0.$$

The optimal control problem

Given the parameters $\theta \in (0,1]$, $\alpha \in (0,1)$, $\varkappa > 0$, $\eta \in (0,1]$, $\rho > 0$, $L > 0$ and $S_0 > 0$, the optimization problem $J(A(\cdot), L^A(\cdot), R_1(\cdot)) \to \max$ can be formulated as the following optimal control problem (P):

$$\dot{A}(t) = A(t)^{\theta}[L^A(t)]^{\eta} R_2(t)^{1-\eta},$$

$$L^A(t) \in [0, L),\ R_1(t) > 0,\ R_2(t) \geq 0, \quad \int_0^{\infty} \big[R_1(t) + R_2(t)\big]\, dt \leq S_0,$$

$$A(0) = A_0 > 0,$$

$$J(A(\cdot), L^A(\cdot), R_1(\cdot)) = \int_0^{\infty} e^{-\rho t} \Big\{ \varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha) \ln R_1(t) \Big\}\, dt \to \max .$$

By an *admissible control* $w(\cdot)\colon [0,\infty) \to \mathbb{R}^3$ in problem (P) we mean a triple $w(\cdot) = (L^A(\cdot), R_1(\cdot), R_2(\cdot))$, $t \geq 0$, of (locally) bounded measurable functions $L^A(\cdot)$, $R_1(\cdot)$ and $R_2(\cdot)$ each of which is defined on the infinite time interval $[0,\infty)$ and satisfies the respective constraints.

Lifting the integral constraint

New state variable $S(t)$, the remaining stock of the exhaustible resource:

$$\dot{S}(t) = -R_1(t) - R_2(t), \qquad S(0) = S_0 > 0.$$

New control variables:

$$u(t) = \frac{R_1(t)}{S(t)}, \qquad v(t) = \frac{R_2(t)}{S(t)}.$$

Optimal control problem:

$$\dot{A}(t) = A(t)^{\theta}[L^A(t)]^{\eta}[v(t)S(t)]^{1-\eta}, \qquad \dot{S}(t) = -[u(t) + v(t)]S(t),$$

$$L^A(t) \in [0, L), \quad u(t) > 0, \quad v(t) \geq 0; \qquad S(t) > 0,$$

$$A(0) = A_0 > 0, \qquad S(0) = S_0 > 0,$$

$$J(A(\cdot), S(\cdot), L^A(\cdot), u(\cdot))$$
$$= \int_0^{\infty} e^{-\rho t}\Big\{\varkappa \ln A(t) + \alpha \ln[L - L^A(t)] + (1-\alpha)\ln[u(t)S(t)]\Big\}\, dt \to \max.$$

It turns out to be possible to prove that $u(\cdot)$ and $v(\cdot)$ are uniformly bounded in any optimal regime, so the state constraint $S(t) > 0$ holds automatically.

Solution method

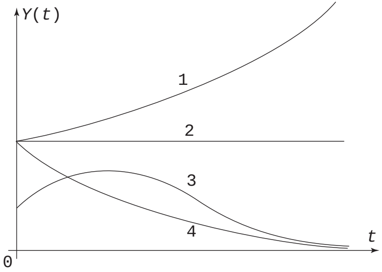
1. Reduction to a one-dimensional problem for the state variable

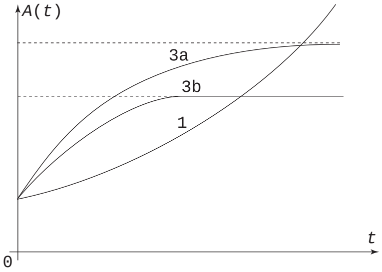
$$x(t) = \frac{S(t)^{1-\eta}}{A(t)^{1-\theta}},$$

$$\dot{x}(t) = -(1-\eta)[u(t)+v(t)]x(t) - (1-\theta)[L^A(t)]^\eta v(t)^{1-\eta} x(t)^2.$$

2. Existence theorem (Balder, 1983).
3. Necessary optimality conditions (PMP) applied to an auxiliary (slightly different) problem, with an additional characterization of the adjoint variable given by the Aseev–Kryazhimskiy (Cauchy-type) formula (Aseev, Kryazhimskiy, 2004–2007).
4. Analysis of the Hamiltonian system of the PMP, which yields a unique optimal solution.

Transitional dynamics of output and knowledge





1. **Balanced (exponential) growth path ($\eta = \theta = 1$, $L\varkappa > \rho(\alpha + \varkappa(1-\alpha))$).**
2. **A zero-growth scenario ($\eta = \theta = 1$, $L\varkappa = \rho(\alpha + \varkappa(1-\alpha))$).**
3. **Rise and decline ($\eta\theta < 1$). Cases** (3*a*)**:** $\eta = 1$ **and** (3*b*)**:** $\eta < 1$.
4. **Decline as population is too small ($\eta = \theta = 1$, $L\varkappa < \rho(\alpha + \varkappa(1-\alpha))$).**

Recall that

$$\dot{A}(t) = A(t)^{\theta}[L^{A}(t)]^{\eta} R_2(t)^{1-\eta}, \qquad \eta \in (0,1], \quad \theta \in (0,1].$$

Random technological jump

Suppose that at time instant $t = T$ the economy jumps to a new technological trajectory (backstop resources):

$$Y(t) = C_1 A(t)^{\varkappa}[L - L^A(t)]^{\alpha'}, \qquad t > T,$$

$$\dot{A}(t) = C_2 A(t)[L^A(t)]^{\eta'}, \qquad t > T.$$

Then the value of the accumulated knowledge stock at instant T can be estimated as

$$V(T, A(T)) = \sup_{L^A(\cdot)} J(T, A(\cdot), L^A(\cdot))$$

where

$$J(T, A(\cdot), L^A(\cdot)) = \int_T^{\infty} e^{-\rho(t-T)}\{\ln[Y(t)]\}\, dt.$$

We find

$$V(T, A(T)) \equiv V(A(T)) = C_3 + \frac{\varkappa'}{\rho} \ln A(T)$$

where $C_3 = C_3(\rho, \varkappa, C_1, C_2, \eta', L, \alpha')$, and obtain the objective functional

$$J(A(\cdot), L^A(\cdot), R_1(\cdot)) = \int_0^T e^{-\rho t}\{\ln[Y(t)]\}\, dt + e^{-\rho T} V(T, A(T)).$$

Modeling the jump

Suppose the probability of a jump in $[t, t+\Delta t]$ under the condition that the jump has not occurred till time t is proportional (in the first approximation) to the number of researches employed in the "production" of knowledge in this time interval:

$$P(t < T < t + \Delta t | t < T) = \nu L^A(t)\Delta t + o(\Delta t).$$

Then the probability density function for the random variable T is

$$\nu L^A(t) e^{-\nu \mathfrak{L}(t)}, \qquad \text{where} \qquad \mathfrak{L}(t) = \int_0^t L^A(s)\, ds \quad \text{for} \;\; 0 \le t < \infty,$$

provided that $\mathfrak{L}(\infty) = \int_0^\infty L^A(s)\, ds = \infty$. If $\mathfrak{L}(\infty) < \infty$, then there is a positive probability that the jump will not occur at all, i.e. $p(T = \infty) = e^{-\nu \mathfrak{L}(\infty)} > 0$.

Maximizing the expectation

The problem is to maximize the expectation of $J(A(\cdot), L^A(\cdot), R_1(\cdot))$ considered as a function of the random variable T:

$$\int_0^\infty \left(\int_0^T e^{-\rho t} \big\{ \ln[Y(t)] \big\} \, dt + e^{-\rho T} V(T, A(T)) \right) \nu L^A(T) e^{-\nu \mathfrak{L}(T)} \, dT$$
$$+ e^{-\nu \mathfrak{L}(\infty)} \int_0^\infty e^{-\rho t} \big\{ \ln[Y(t)] \big\} \, dt \to \max .$$

After some transformations, we reduce the problem to the following equivalent one:

$$\int_0^\infty e^{-\rho t - \nu \mathfrak{L}(t)} \bigg\{ \frac{\varkappa}{\rho} L^A(t)^\eta v(t)^{1-\eta} x(t) + \alpha \ln(L - L^A(t)) + (1-\alpha) \ln u(t)$$
$$+ (1-\alpha) \ln S(t) - C_3 \rho \bigg\} \, dt \to \max,$$

where

$$x(t) = \frac{S(t)^{1-\eta}}{A(t)^{1-\theta}} > 0, \qquad u(t) = \frac{R_1(t)}{S(t)}, \qquad v(t) = \frac{R_2(t)}{S(t)}, \qquad t > 0.$$

Formulation of the optimal control problem

Thus we come to the following infinite-horizon optimal control problem (P′) with discount:

$$\dot{x}(t) = -(1-\eta)[u(t)+v(t)]x(t) - (1-\theta)L^A(t)^\eta v(t)^{1-\eta}x(t)^2,$$

$$x(0) = x_0 := S_0^{1-\eta}/A_0^{1-\theta},$$

$$\dot{S}(t) = -[u(t)+v(t)]S(t), \qquad S(0) = S_0,$$

$$\dot{B}(t) = -\nu L^A(t)B(t), \qquad B(0) = 1,$$

$$L^A(t) \in [0, L), \qquad u(t) > 0, \qquad v(t) \geq 0; \qquad S(t) > 0,$$

$$\int_0^\infty e^{-\rho t} B(t)\Big\{ \frac{\varkappa}{\rho} L^A(t)^\eta v(t)^{1-\eta} x(t) + \alpha \ln(L - L^A(t))$$

$$+ (1-\alpha)\ln u(t) + (1-\alpha)\ln S(t) - C_3\rho \Big\}\, dt \to \max.$$

Solution method

1. Optimal policies are uniformly bounded: $u(\cdot) \leq U_0$ and $v(\cdot) \leq V_0$, so the constraint $S(t) > 0$ holds automatically.
2. Existence theorem (Balder, 1983).
3. Necessary optimality conditions (PMP) with an additional characterization of the adjoint variable given by the Aseev–Kryazhimskiy (Cauchy-type) formula (K.O. Besov, Proc. Steklov Inst. Math. **284**, 2014, pp. 50–80). This version of the PMP extends the earlier results of Aseev, Kryazhimskii (2004–2007) and Aseev, Kryazhimskii, Besov (2012) to the class of problems with locally unbounded instantaneous utility function.
4. Analysis of the (six-dimensional) Hamiltonian system of the PMP (for three state variables $x(\cdot)$, $S(\cdot)$ and $B(\cdot)$ and three adjoint variables), with three initial conditions for the state variables and three additional Cauchy-type relations for the adjoint variables.

Transversality conditions at infinity

Remark. This is one of the main difficulties in infinite-horizon optimal control problems — the absence of boundary conditions for the adjoint variables (transversality conditions at infinity) in the general case. In some more specific cases, there is a good substitution for the transversality conditions at infinity, namely, a Cauchy-type formula for the adjoint variables, which was first proved by Aseev and Kryazhimskii in 2004 and which was later shown to hold in more general cases.

In particular, in (K.O. Besov, 2014) this formula was shown to hold in the so-called dominating-discount case for a certain class of problems with locally unbounded instantaneous utility function. It is this result that we apply to problem (P') (earlier versions of the PMP cannot be direcly applied to (P')).

Conclusions

Recall that for the deterministic problem (P) (without technological jump), in the optimal regime we have

$$\begin{aligned} u(t) &\to \rho \qquad \text{(Hotelling's law)}, \\ v(t) &\to 0, \\ L^A(t) &\to 0 \qquad \text{as} \quad t \to \infty. \end{aligned}$$

For problem (P′) in the optimal regime we have

$$\begin{aligned} u(t) &\to \rho + \nu L \qquad \text{(Hotelling's law)}, \\ v(t) &\to 0, \\ L^A(t) &\to L \qquad \text{as} \quad t \to \infty. \end{aligned}$$

The latter relation shows that the number of researches should increase in time (since $L^A(t)$ is always less than L).

PMP: general formulation

Optimal control problem (P):

$$\dot{x}(t) = f(x(t), u(t)), \qquad u(t) \in U, \tag{1}$$

$$x(0) = x_0, \tag{2}$$

$$J(x(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t} g(x(t), u(t))\, dt \to \max, \tag{3}$$

where $x_0, x(t) \in G \subset \mathbb{R}^n$; G is open; $U \subset \mathbb{R}^m$ is a nonempty bounded set s.t. $\overline{U} \setminus U$ is closed; $f, f_x \in C(G \times \overline{U})$ and $g, g_x \in C(G \times U)$.

($\mathbf{A}_g$) $g(x, u) \to -\infty$ as $G \times U \ni (x, u) \to (\bar{x}, \bar{u}) \in G \times (\overline{U} \setminus U)$, and

$$\|g_x(x, u)\| \le a(x)(|g(x, u)| + 1), \qquad (x, u) \in G \times U, \tag{4}$$

where $a\colon G \to \mathbb{R}$ is continuous.

(A1) All admissible trajectories $x(\cdot)$ of the extended system ((1) with $u(t) \in \overline{U}$) are extendable to $[0, \infty)$.

(A2) For any $x \in G$ the following set is convex:

$$Q(x) = \{(z^0, z) \in \mathbb{R}^{n+1} \colon\ z^0 \leq g(x,u),\ z = f(x,u),\ u \in U\}.$$

(A3) For any admissible pair $(x(\cdot), u(\cdot))$,

$$\int_T^{T'} e^{-\rho t} g(x(t), u(t))\, dt \leq \omega(T), \qquad 0 \leq T \leq T',$$

where $\omega\colon [0,\infty) \to \mathbb{R}_+$ and $\omega(t) \to +0$ as $t \to \infty$.

Let $Y_{(x,u)}$ and $Z_{(x,u)}$ be the fundamental matrices of the linear systems

$$\dot{y}(t) = f_x(x(t), u(t)) y(t), \qquad \dot{z}(t) = -[f_x(x(t), u(t))]^* z(t), \qquad t \geq 0.$$

(A4) For any admissible pair $(x(\cdot), u(\cdot))$ and all $0 \leq T \leq T'$,

$$\left\| \int_T^{T'} e^{-\rho s} [Y_{(x,u)}(s)]^* g_x(x(s), u(s))\, ds \right\| \leq \tilde{\omega}(T) + C \left| \int_T^{T'} e^{-\rho s} g(x(s), u(s))\, ds \right|,$$

where $\tilde{\omega}\colon [0,\infty) \to \mathbb{R}$ with $\tilde{\omega}(t) \to +0$ as $t \to \infty$ and $C = \mathrm{const} \geq 0$.

Theorem

If conditions (A_g), (A1)–(A4) hold and $(x_*(\cdot), u_*(\cdot))$ is an optimal admissible pair in problem (P), then

1) the following improper integral converges for any $t \geq 0$:

$$I_*(t) = \int_t^\infty e^{-\rho s}[Y_{(x_*,u_*)}(s)]^* g_x(x_*(s), u_*(s))\, ds;$$

2) the function $\psi(t) = Z_{(x_*,u_*)}(t) I_*(t)$, together with $(x_*(\cdot), u_*(\cdot))$, satisfies the core relations of the PMP (in the normal form):

$$\dot{\psi}(t) = -\mathcal{H}_x(x_*(t), t, u_*(t), \psi(t), 1),$$

$$\mathcal{H}(x_*(t), t, u_*(t), \psi(t), 1) \overset{\text{a.e.}}{=} \sup_{u \in U} \mathcal{H}(x_*(t), t, u, \psi(t), 1);$$

3) the stationarity condition holds (in the normal form):

$$\mathcal{H}(x_*(t), t, u_*(t), \psi(t), 1) = \rho \int_t^\infty e^{-\rho s} g(x_*(s), u_*(s))\, ds, \quad t \geq 0.$$

Hamilton–Pontryagin function $\mathcal{H}(x, t, u, \psi, \psi^0) = \langle f(x,u), \psi \rangle + \psi^0 e^{-\rho t} g(x,u)$.

Publications

1. S. Aseev, K. Besov, S. Kaniovski, “The problem of optimal endogenous growth with exhaustible resources revisited,” in *Green Growth and Sustainable Development* (Springer, Berlin, 2013), Dynamic Modeling and Econometrics in Economics and Finance **14**, pp. 3–30.
2. S.M. Aseev, K.O. Besov, A.V. Kryazhimskiy, “Infinite-horizon optimal control problems in economics,” Russ. Math. Surv. **67**, 195–253 (2012).
3. K.O. Besov, “On necessary optimality conditions for infinite-horizon economic growth problems with locally unbounded instantaneous utility function,” Proc. Steklov Inst. Math. **284**, 50–80 (2014).

THANK YOU!