CLASSIFICATION OF BINARY FORMS WITH CONTROL PARAMETER

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Problem

Let $V_n(u)$ be the space of *binary forms*, whose coefficients depend on the *control parameter*:

$$f(x,y;u) = \sum_{i=0}^{n} a_i(u) x^i y^{n-i}, \quad \text{where } a_i \text{ are holomorphic functions.}$$

The pseudogroup $G := \mathrm{SL}_2 \leftthreetimes (\mathcal{F}(u) \times \mathrm{T}(u))$ acts on $V_n(u)$:

1) «semisimple part» SL_2:

$$\mathrm{SL}_2 \ni A \colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A^{-1} \begin{pmatrix} x \\ y \end{pmatrix};$$

2) «functional part» $\mathcal{F}(u)$:

$$u \mapsto \varphi(u);$$

3) «torus» $\mathrm{T}(u)$:

$$f \mapsto \lambda(u) f.$$

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When are two given binary forms f and $\widetilde{f}$ with control parameter G-equivalent?

Geometric interpretation:

$$\text{binary form } f \quad \leftrightsquigarrow \quad \text{set of projective points}$$

$$f(x,y;u) = (\alpha_1(u)y - \beta_1(u)x)\cdot\ldots\cdot(\alpha_n(u)y - \beta_n(u)x)$$

$\Rightarrow$ the set of zeros of f is the non–ordered set

$$\{P_1(u),\ldots,P_n(u)\}$$

of projective points.

Problem

Classify non-ordered sets of complex projective points with control parameter on the line with respect to projective transformations

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Let $a_i(u) \equiv a_i = \mathrm{const}$
$\Rightarrow$ we get famous algebraic

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When are two given binary forms f and $\widetilde{f}$ of degree n with complex coefficients SL_2-equivalent?

History:

- Bool (1841) — $n = 3$;
- Cayley, Eisenstein (1851) — $n = 4$; debut of classical invariant theory;
- Cayley, Hermite (1860) — $n = 5$;
- Gordan, Shioda, Hilbert, Dixmier, Lazard, etc. (1980–2000) — $n \leqslant 10$, $n = 12$;
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Definitions and notations

Let $\mathbb{C}^3$ be complex space with coordinates (x, y, u).

k-jet $[f]_b^k$ of function $f\colon \mathbb{C}^3 \to \mathbb{C}$ is a segment of the Teylor series of function f in point b up to the members of order k.

k-jet space J^k is the set of all k-jets for all functions in all points.

Canonical coordinates:

$$(x, y, u, h, h_x, h_y, h_u, h_{xx}, h_{xy}, h_{yy}, h_{xu}, h_{yu}, h_{uu} \dots),$$

$h_x([f]_b^k) = f_x(b)$, $h_{xy}([f]_b^k) = f_{xy}(b)$, etc.

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Differential Euler equation

Binary forms with control parameter can be considered as solutions of the *Euler differential equation*

$$\mathcal{E} := \{x h_x + y h_y = n h\} \subset J^1.$$

Prolongation of the Euler equation:

$$\mathcal{E}^{(1)} = \left\{\mathcal{E}, \frac{d}{dx}\mathcal{E}, \frac{d}{dy}\mathcal{E}, \frac{d}{du}\mathcal{E}\right\} \subset J^2,$$

where $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{du}$ are *operators of total derivations*:

$$\frac{d}{dx} = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h} + h_{xx} \frac{\partial}{\partial h_x} + h_{xy} \frac{\partial}{\partial h_y} + h_{xu} \frac{\partial}{\partial h_u} + \dots$$

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Differential invariants

The action of the pseudogroup G on 0-jet space J^0 prolongs to the action on all prolongations $\mathcal{E}^{(k-1)} \subset J^k$:

$$g \circ [f]_b^k = [g \circ f]_{gb}^k.$$

Definition

- *Differential invariant of the action of pseudogroup G of order k* is G-invariant function on manifold $\mathcal{E}^{(k-1)}$, which is polynomial in derivatives h_σ, h^{-1} and $(h_x h_{yu} - h_y h_{xu})^{-1}$.
- *Invariant derivation* is a combination of total derivations, which commutes with the action of group G.

Remark. Function $h_x h_{yu} - h_y h_{xu}$ is «total Poisson bracket» $\{h, h_u\}$.

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Algebra of differential invariants

Theorem

Differential invariant algebra of the action of pseudogroup G is freely generated by differential invariant

$$H := \frac{h_{xx}h_{yy} - h_{xy}^2}{h^2}$$

of order 2 and by invariant derivations

$$\nabla_1 := \frac{h_y}{h}\frac{d}{dx} - \frac{h_x}{h}\frac{d}{dy} \quad \text{and} \quad \nabla_2 := \frac{h^2}{h_x h_{yu} - h_y h_{xu}} \cdot \frac{d}{du}.$$

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Classification of binary forms

Definition

Binary form $f \in V_n(u)$ is said to be regular, if the restrictions of the invariants H, H_1 and H_2 on form f are functionally independent in points of some domain $\Omega \subset \mathbb{C}^3$.

Consider the regular binary form f. Then the restrictions of invariants H_{11}, H_{12} and H_{22} on form f can be extended through the restrictions of the invariants H, H_1 and H_2 on f:

$$H_{11} = A(H, H_1, H_2), \quad H_{12} = B(H, H_1, H_2), \quad H_{22} = C(H, H_1, H_2).$$

The triple (A, B, C) is said to be *triple of dependencies* of form f.

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Consider the regular binary form f. Then the restrictions of invariants H_{11}, H_{12} and H_{22} on form f can be extended through the restrictions of the invariants H, H_1 and H_2 on f:

$$H_{11} = A(H, H_1, H_2), \;\; H_{12} = B(H, H_1, H_2), \;\; H_{22} = C(H, H_1, H_2).$$

The triple (A, B, C) is said to be *triple of dependencies* of form f.

Classification of binary forms

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Classification theorem

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Proof

"$\Leftarrow$" Let $(A, B, C) = (\widetilde{A}, \widetilde{B}, \widetilde{C})$ for two regular forms f and $\widetilde{f}$.

Consider "invariant coordinate systems"

$$S := (H(f), H_1(f), H_2(f)), \;\; \widetilde{S} := (H(\widetilde{f}), H_1(\widetilde{f}), H_2(\widetilde{f})).$$

Let us take two jets

$$[f]_a^4 \quad \text{and} \quad [\widetilde{f}]_b^4$$

with the same coordinates in S and $\widetilde{S}$ correspondingly.

Values of all differential invariants of the 4-th order in these jets coincide. Hence, these jets are G-equivalent, and

$$\exists\, g \in G : \quad g(a) = b \quad \text{and} \quad [g \circ f]_b^4 = [\widetilde{f}]_b^4.$$

After prolongation we get $[g \circ f]_b^\infty = [\widetilde{f}]_b^\infty$, hence, $g \circ f = \widetilde{f}$. □

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