SELF–ADJOINT COMMUTING DIFFERENTIAL OPERATORS OF RANK 2 AND THEIR DEFORMATIONS GIVEN BY THE SOLITON EQUATIONS

Valentina Davletshina

Novosibirsk State University, Sobolev Institute of Mathematics

International Youth Conference "Geometry and Control"
Moscow, 17.04.2014

Scheme of the talk

1. deformations of commutative rings of self-adjoint ordinary differential operators of rank two.
2. some examples of self-adjoint ordinary differential operators of 4 and 4g+2 orders and the rigorous proof of the fact that these operators are operators of the rank two.

Problem 1

We study rank two solutions of the following system

$$\mathbf{V_t = \frac{1}{4}\Big(6VV_x + 6W_x + V_{xxx}\Big), \quad W_t = \frac{1}{2}\Big(-3VW_x - W_{xxx}\Big).} \quad \mathbf{(1)}$$

This system is equivalent to the condition of commutation of self–adjoint operator of order **4** and skew–symmetric operator of order **3**:

$$\mathbf{[L_4, \partial_t - A] = 0,} \quad \mathbf{(2)}$$

where

$$\mathbf{L_4 = \Big(\partial_x^2 + V(x,t)\Big)^2 + W(x,t), \; A = \partial_x^3 + \tfrac{3}{2}V(x,t)\,\partial_x + \tfrac{3}{4}V_x(x,t).}$$

Rank two means that the condition (2) must hold as well as the condition

$$\mathbf{[L_4, L_{4g+2}] = 0.}$$

Then the common eigenfunctions of $\mathbf{L_4}$ and $\mathbf{L_{4g+2}}$ form a vector bundle of rank two. In general case the problem of finding of commuting operators of rank two is not solved. If orders are coprime, then the common eigenfunctions can be found explicitly. We construct solutions of the system (1). The behavior of solutions looks like g-soliton KdV equation.

$$L_n = \sum_{i=0}^{n} v_i(x)\partial_x^i, \quad L_m = \sum_{j=0}^{m} u_j(x)\partial_x^j, \quad L_r = \sum_{k=0}^{r} w_k(x)\partial_x^k.$$

Lemma (Schur, 1905)

If $L_nL_m = L_mL_n$ and $L_nL_r = L_rL_n$ ($L_n \neq \mathrm{const}$), then $L_mL_r = L_rL_m$.

$$L_n = \sum_{i=0}^{n} v_i(x)\partial_x^i, \quad L_m = \sum_{j=0}^{m} u_j(x)\partial_x^j, \quad L_r = \sum_{k=0}^{r} w_k(x)\partial_x^k.$$

Lemma (Schur, 1905)

If $L_nL_m = L_mL_n$ and $L_nL_r = L_rL_n$ ($L_n \neq \mathrm{const}$), then $L_mL_r = L_rL_m$.

Lemma (Burchnall, Chaundy, 1923)

If $L_nL_m = L_mL_n$, then there exists a non-trivial polynomial $Q(\lambda, \mu)$ of two commuting variables such that $Q(L_n, L_m) = 0$.

$$L_n = \sum_{i=0}^{n} v_i(x)\partial_x^i, \quad L_m = \sum_{j=0}^{m} u_j(x)\partial_x^j, \quad L_r = \sum_{k=0}^{r} w_k(x)\partial_x^k.$$

Lemma (Schur, 1905)

If $L_n L_m = L_m L_n$ and $L_n L_r = L_r L_n$ ($L_n \neq \mathrm{const}$), then $L_m L_r = L_r L_m$.

Lemma (Burchnall, Chaundy, 1923)

If $L_n L_m = L_m L_n$, then there exists a non-trivial polynomial $Q(\lambda, \mu)$ of two commuting variables such that $Q(L_n, L_m) = 0$.

Example

$$L_2 = \frac{d^2}{dx^2} - \frac{2}{x^2}, \quad L_3 = \frac{d^3}{dx^3} - \frac{3}{x^2}\frac{d}{dx} + \frac{3}{x^3}$$

$$L_2^3 = L_3^2, \quad Q(\lambda, \mu) = \lambda^3 - \mu^2.$$

Definition

A *spectral curve* Γ of a pair L_n, L_m is defined by the equation $Q = 0$

$$\Gamma = \{(\lambda, \mu) \in \mathbb{C}^2 : Q(\lambda, \mu) = 0\}.$$

If ψ is a common eigenfunction of the operators L_n and L_m

$$L_n\psi = \lambda\psi, \quad L_m\psi = \mu\psi,$$

then $(\lambda, \mu) \in \Gamma$.

Definition

A *spectral curve* Γ of a pair L_n, L_m is defined by the equation $Q = 0$

$$\Gamma = \{(\lambda, \mu) \in \mathbb{C}^2 : Q(\lambda, \mu) = 0\}.$$

If ψ is a common eigenfunction of the operators L_n and L_m

$$L_n\psi = \lambda\psi, \quad L_m\psi = \mu\psi,$$

then $(\lambda, \mu) \in \Gamma$.

Definition

For almost all $(\lambda, \mu) \in \Gamma$ the dimension of the space of common eigenfunctions of L_n and L_m is called the *rank*

$$\ell = \dim_{\mathbb{C}}\{\psi : L_n\psi = \lambda\psi,\ L_m\psi = \mu\psi\}.$$

The dimension is equal to the greatest common divisor of n and m.

History

There is only a classification of commutative rings of ordinary differential operators of any high rank obtained by Krichever but in general case such operators are still not found. Krichever and Novikov have found operators of rank two corresponding to elliptic spectral curve. Mokhov has found operators of rank three corresponding to elliptic spectral curve as well.

We consider a pair $\mathbf{L_4}$, $\mathbf{L_{4g+2}}$ of commuting differential operators of rank two. The dimension of space of common eigenfunctions of $\mathbf{L_4}$ and $\mathbf{L_{4g+2}}$ commuting with each other with common eigenvalues $\mathbf{z}$ and $\mathbf{w}$ fixed is equal to 2

$$\mathbf{dim}_{\mathbb{C}} \{\psi : \mathbf{L_4}\psi = \mathbf{z}\psi, \mathbf{L_{4g+2}}\psi = \mathbf{w}\psi\} = \mathbf{2}.$$

The common eigenfunction ψ of operators $\mathbf{L_4}$ and $\mathbf{L_{4g+2}}$ is a vector Baker–Akhiezer function.

We consider a pair $\mathbf{L_4}$, $\mathbf{L_{4g+2}}$ of commuting differential operators of rank two. The dimension of space of common eigenfunctions of $\mathbf{L_4}$ and $\mathbf{L_{4g+2}}$ commuting with each other with common eigenvalues $\mathbf{z}$ and $\mathbf{w}$ fixed is equal to 2

$$\mathbf{dim}_{\mathbb{C}} \{\psi : \mathbf{L_4}\psi = \mathbf{z}\psi, \mathbf{L_{4g+2}}\psi = \mathbf{w}\psi\} = \mathbf{2}.$$

The common eigenfunction ψ of operators $\mathbf{L_4}$ and $\mathbf{L_{4g+2}}$ is a vector Baker–Akhiezer function.

The vector Baker–Akhiezer function can be constructed by spectral data:

$$\left\{\mathbf{\Gamma}, \mathbf{q}, \mathbf{k}^{-1}, \gamma, \alpha\right\},$$

where $\mathbf{\Gamma}$ is an algebraic curve; $\mathbf{q} \in \mathbf{\Gamma}$; $\mathbf{k}^{-1}$ is a local parameter nearby $\mathbf{q}$, $\mathbf{k}^{-1}(\mathbf{q}) = \mathbf{0}$; $\gamma = \gamma_1 + \ldots + \gamma_{2g}$; α is a set of vectors $(1, \alpha_1), \ldots, (1, \alpha_{2g})$.

Definition

The *vector Baker–Akhiezer function* $\psi = (\psi_1, \psi_2)$ is defined by the following properties:

1. In the neighborhood of $\mathbf{q}$ the vector–function ψ has the form

$$\psi(x, P) = \sum_{s=0}^{\infty} \left(\xi_s(x) k^{-s}\right) \Psi_0(x, P),$$

   where $\xi_0 = (1, 0)$. The matrix $\Psi_0(x, P)$ satisfies the equation

$$\frac{d}{dx}\Psi_0 = A\Psi_0, \quad A = \begin{pmatrix} 0 & 1 \\ k + u(x) & 0 \end{pmatrix}$$

2. The components of ψ are meromorphic functions on $\Gamma \setminus \{q\}$ with the simple poles in $\gamma_1, \ldots, \gamma_{2g}$, and

$$\mathrm{Res}_{\gamma_i} \psi_1 = \alpha_i \, \mathrm{Res}_{\gamma_i} \psi_2, \qquad 1 \le i \le 2g.$$

Common eigenfunctions of L_4 and L_{4g+2} satisfy the second order differential equation

$$\psi''(x,P) = \chi_1(x,P)\psi'(x,P) + \chi_0(x,P)\psi(x,P).$$

The coefficients $\chi_0(x,P)$, $\chi_1(x,P)$ are rational functions on Γ with $2g$ simple poles. χ_0 has also an additional simple pole at infinity.
Let $k-\gamma_i(x)$ be a local parameter nearby $P_1(x),\dots,P_{2g}(x)$. Then

$$\chi_0(x,P) = \frac{-v_{i,0}(x)\gamma_i'(x)}{k-\gamma_i(x)} + d_{i,0}(x) + O\big(k-\gamma_i(x)\big),$$

$$\chi_1(x,P) = \frac{-\gamma_i'(x)}{k-\gamma_i(x)} + d_{i,1}(x) + O\big(k-\gamma_i(x)\big),$$

Theorem (I. M. Krichever, S. P. Novikov, 1978)

$$d_{i,0}(x) = v_{i,0}^2(x) + v_{i,0}(x)d_{i,1}(x) - v_{i,0}'(x).$$

Let L_4 and L_{4g+2} be a pair of commuting differential operators of rank two. Then

$$\Gamma : w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \ldots + c_0.$$

The curve Γ has a holomorphic involution

$$\sigma : \Gamma \to \Gamma, \quad \sigma(z, w) = (z, -w).$$

Common eigenfunctions of L_4 and L_{4g+2} satisfy the second order differential equation

$$\psi''(x, P) = \chi_1(x, P)\,\psi'(x, P) + \chi_0(x, P)\,\psi(x, P),$$

where χ_s is meromorphic on Γ with the simple poles depending on x, χ_0 has also an additional simple pole at infinity. These functions satisfy Krichever's equations. For finding operators L_4 and L_{4g+2} it is enough to find χ_0 and χ_1.

Theorem

The operator L_4 is self–adjoint if and only if

$$\chi_1(x, P) = \chi_1(x, \sigma P).$$

Theorem

The operator $\mathbf{L_4}$ is self–adjoint if and only if

$$\chi_1(x, P) = \chi_1(x, \sigma P).$$

Theorem

If $\mathbf{L_4}$ is self-adjoint, $\mathbf{L_4} = (\partial_x^2 + V(x))^2 + W(x)$, then

$$\chi_0 = -\frac{Q_{xx}}{2Q} + \frac{w}{Q} - V, \qquad \chi_1 = \frac{Q_x}{Q},$$

where

$$Q = z^g + \alpha_{g-1}(x, t)z^{g-1} + \ldots + \alpha_0(x, t).$$

Function $\mathbf{Q}$ satisfies the equation

$$4F_g(z) = 4(z - W)Q^2 - 4V(Q_x)^2 + (Q_{xx})^2 - 2Q_xQ_{xxx} +$$

$$2Q(2V_xQ_x + 4VQ_{xx} + Q_{xxxx}). \qquad (3)$$

With the help of the equation **(3)**, the first examples of self-adjoint commuting differential operators of rank two corresponding to a spectral curve of higher genus were constructed.

Example

$$L_4^{\sharp} = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + \alpha_3 g(g+1)x$$

$$L_4^{\natural} = (\partial_x^2 + \alpha_1 \cosh(x) + \alpha_0)^2 + \alpha_1 g(g+1)\cosh(x), \quad \alpha_1 \neq 0$$

$$L_4^{\flat} = (\partial_x^2 + \alpha_1 e^x + \alpha_0)^2 + \alpha_1 g(g+1)e^x, \quad \alpha_1 \neq 0,$$

We study dynamics of polynomial $\mathbf{Q}$ provided that $\mathbf{V}$ and $\mathbf{W}$ satisfy **(1)**.

Theorem (D, 2013)

Let $\mathbf{L_4} = (\partial_x^2 + \mathbf{V(x,t)})^2 + \mathbf{W(x,t)}$ be self-adjoint and $[\mathbf{L_4}, \partial_t - \mathbf{A}] = \mathbf{0}$, i.e.

$$\mathbf{V_t} = \frac{1}{4}\Big(6\mathbf{V}\mathbf{V_x} + 6\mathbf{W_x} + \mathbf{V_{xxx}}\Big), \quad \mathbf{W_t} = \frac{1}{2}\Big(-3\mathbf{V}\mathbf{W_x} - \mathbf{W_{xxx}}\Big). \quad \mathbf{(1)}$$

Suppose that $[\mathbf{L_4}, \mathbf{L_{4g+2}}] = \mathbf{0}$, then function $\mathbf{Q}$ satisfies the equation

$$\mathbf{Q_t} = \frac{1}{2}\Big(-3\mathbf{V}\mathbf{Q_x} - \mathbf{Q_{xxx}}\Big).$$

This equation yields symmetry of equation

$$4\mathbf{F_g(z)} = 4(\mathbf{z} - \mathbf{W})\mathbf{Q}^2 - 4\mathbf{V}(\mathbf{Q_x})^2 + (\mathbf{Q_{xx}})^2 - 2\mathbf{Q_x}\mathbf{Q_{xxx}} +$$

$$2\mathbf{Q}(2\mathbf{V_x}\mathbf{Q_x} + 4\mathbf{V}\mathbf{Q_{xx}} + \mathbf{Q_{xxxx}}).$$

Remark

If we substitute the operator **A** by a skew–symmetric operator of order $2n+1$, then we obtain the evolution equation for **Q**. For example,

if $n=2$, then

$$Q_{t_2} = \frac{1}{8}\big(-4QW_x + 2V_xQ_{xx} + Q_x(8z - 5V^2 + 2W - V_{xx}) - 2VQ_{xxx}\big),$$

if $n=3$, then

$$Q_{t_3} = \frac{1}{32}\Big(-14V^3Q_x - 2V\left(-6(2QW_x + V_xQ_{xx}) + Q_x\,(24z + 18W + 5V_{xx})\right) - 6V^2Q_{xxx} + 2\left(6W_xQ_{xx} - (8z + 6W + V_{xx})Q_{xxx} + Q_{xx}V_{xxx} + 4QW_{xxx}\right) - Q_x\,(7V_x^2 + 10W_{xx} + V_{xxxx})\Big).$$

These equations give symmetry of equation

$$4F_g(z) = 4(z - W)Q^2 - 4V(Q_x)^2 + (Q_{xx})^2 - 2Q_xQ_{xxx} + 2Q(2V_xQ_x + 4VQ_{xx} + Q_{xxxx}).$$

If $\mathbf{g=1}$, then the equation

$$4F_g(z) = 4(z-W)Q^2 - 4V(Q_x)^2 + (Q_{xx})^2 - 2Q_xQ_{xxx} + 2Q(2V_xQ_x + 4VQ_{xx} + Q_{xxxx})$$

is reduced to Krichever–Novikov equation.
And equations for polynomials $\mathbf{Q_{t_2}}$ and $\mathbf{Q_{t_3}}$ are reduced to evolution Krichever–Novikov equation.

$$W_{t_1} = \frac{48F_1\left(\frac{1}{2}(-c_2 - W)\right) - W_{xx}^2 + 2W_xW_{xxx}}{8W_x},$$

$$W_{t_2} = \frac{1}{128W_x^3}\Big(-1280F_1^2(\gamma) - 16\big(4c_2 + 2W - F_{1xx}(\gamma)\big)W_x^4 - 45W_{xx}^4 + 100W_xW_{xx}^2W_{xxx} + 160F_1(\gamma)\left(5W_{xx}^2 - 2W_xW_{xxx}\right) + 20W_x^2\big(-W_{xxx}^2 + 2W_{xx}(4F_{1x}(\gamma) - \partial_x^4 W)\big) + 8W_x^3\partial_x^5 W\Big),$$

where $\gamma = -\frac{1}{2}(c_2 + W)$.

Theorem (D, 2013)

For $\mathbf{n = 1, 2, 3}$ conditions of commutation $[\mathbf{L_4}, \partial_{\mathbf{t_n}} - \mathbf{A_{t_n}}] = \mathbf{0}$ have next solution:

$$V(x, t_1) = -10\wp(at_1 + x) - \frac{2a}{21},$$

$$W(x, t_2) = -40\wp^2(at_1 + x) - \frac{20}{21}(8a + 7g_2)\wp(at_1 + x),$$

$$V(x, t_2) = -8\wp(at_2 + x) - \frac{2g_2}{3}.$$

$$W(x, t_2) = 24\wp^2(at_2 + x) + 4g_2\wp(at_2 + x) + \frac{4a}{5},$$

$$V(x, t_3) = -10\wp(at_3 + x) + b.$$

$$W(x, t_3) = -40\wp^2(at_3 + x) + 8b\wp(at_3 + x),$$

$\mathbf{a}$, $\mathbf{b}$ are constants, $\wp(\mathbf{z})$ is the Weierstrass elliptic function satisfying the equation $(\wp'(z))^2 = 4\wp^3(z) + g_2\wp^2(z) + g_1\wp(z) + g_0$.

Problem 2

Example

$$L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + \alpha_3 g(g+1)x$$

$$L_4^\natural = (\partial_x^2 + \alpha_1 \cosh(x) + \alpha_0)^2 + \alpha_1 g(g+1)\cosh(x), \quad \alpha_1 \neq 0$$

$$L_4^\flat = (\partial_x^2 + \alpha_1 e^x + \alpha_0)^2 + \alpha_1 g(g+1)e^x, \quad \alpha_1 \neq 0,$$

Operator $\mathbf{L_4}$ is known to commute with operator $\mathbf{L_{4g+2}}$ but formally it is not clear if these operators commute with odd order operators or not.

Theorem (D, Shamaev, 2014)

The operator $\mathbf{L}_4^\sharp$ does not commute with any differential operator of odd order.

Theorem (D, Shamaev, 2014)

The operator $\mathbf{L}_4^\natural$ does not commute with any differential operator of odd order.

Theorem (D, Shamaev, 2014)

The operator $\mathbf{L}_4^\flat$ does not commute with any differential operator of odd order.

These theorems rigorously prove that $\mathbf{L}_4^\sharp$ and $\mathbf{L}_{4g+2}^\sharp$, $\mathbf{L}_4^\natural$ and $\mathbf{L}_{4g+2}^\natural$, $\mathbf{L}_4^\flat$ and $\mathbf{L}_{4g+2}^\flat$ are differential operators of rank two.

Corollary

1. Solutions of rank two are constructed.
2. The rigorous proof of the fact that considered examples of operators are operators of the rank two was obtained.

THANK YOU
FOR ATTENTION!