MTW vs convexity of injectivity domains

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Monge's problem

Settings

- space: $\mathbb{R}^n$, or M a smooth compact connected Riemannian manifold.
- cost $c : X \times Y \to \mathbb{R}$, in particular $\frac{1}{2}d^2(x,y)$.
- two probability measures $\mu = f\mathrm{Leb}$, $\nu = g\mathrm{Leb}$.
- $\mathbf{t}_{\#}\mu = \nu$: for any measurable set B we have $\nu(B) = \mu\left(\mathbf{t}^{-1}(B)\right)$.

Problem (Monge 1781)

Find T such that

$$T = \mathrm{argmin}_{\mathbf{t}_{\#}\mu=\nu}\left(\int_{\mathrm{M}} \mathrm{c}\left(\mathrm{x}, \mathbf{t}(\mathrm{x})\right) \mathrm{d}\mu(\mathrm{x})\right).$$

Riemannian vocabulary

$x \in M$, $v \in T_x M$

- The tangent space: $T_x M$.
- d: $M \times M \to \mathbb{R}$ the geodesic distance.
- The exponential map $(x, v) \mapsto \exp_x(v) = \gamma(x, v, 1)$ where $\gamma : TM \times I \to \mathbb{R}$ is the unique geodesic path with initial position x and initial speed v.

Important domains

- The injectivity domain

$$\mathrm{I}(x) = \{v \in T_x M \,|\, \exists\, t > 1 \text{ s.t. } d(x, \exp_x(tv)) = |tv|_x\},$$

- The non focal domain

$$\mathrm{NF}(x) = \Big\{v \in T_x M \,|\, d_{tv}\exp_x \text{ is not singular for any } t \in [0,1]\Big\}.$$

- Starshaped domains, ρ is the radial distance.

Existence, regularity

- $\left(\mathbb{R}^n, \frac{1}{2}|x-y|^2\right)$, $T(x) = x + \nabla\phi(x) = \nabla\varphi(x)$, φ convex.
  BRENIER '87
- $\left(M, \frac{1}{2}d^2(x,y)\right)$, $T(x) = \exp_x(\nabla\phi(x))$, ϕ $\frac{d^2}{2}$-convex.
  MCCANN '01
- T is solution of the Monge-Ampère equation

$$\det\left(\nabla^2\phi(x) + \nabla^2_{xx}c\,(x, T(x))\right) = \frac{f(x)}{g(T(x))}\left|\det \nabla_{x,y}c\,(x, T(x))\right|.$$

Problem

Given f, g C^0, C^∞, T is C^1, C^∞ ?

Ma-Trundinger-Wang tensor

- The Ma-Trudinger-Wang tensor (MTW) first appearant. In $(\mathbb{R}^n, c(x,y))$, for $\langle \xi . \eta \rangle = 0$,

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2} \sum_{i,j,k,l,p,q,r,s} \left(c^{p,q} c_{ij,p} c_{q,rs} - c_{ij,rs}\right) c^{r,k} c^{s,l} \xi_i \xi_j \eta_k \eta_l .$$

- In $\mathbb{R}^n$, if this quantity is always non negative then T is smooth.
- The Ma-Trudinger-Wang tensor in the Riemannian case.

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2} \left. \frac{d^2}{ds^2} \right|_{s=0} \left. \frac{d^2}{dt^2} \right|_{t=0} \frac{d^2}{2} \Big(\exp_x(t\xi), \exp_x(v+s\eta) \Big).$$

MTW conditions

M satisfies $\mathrm{MTW(K, C)}$ if for all $(x, v) \in M \times I(x)$ and $(\xi, \eta) \in T_x M \times T_x M$,

$$\mathfrak{S}_{(x,v)}(\xi, \eta) \geq K|\xi|^2|\eta|^2 - C|\langle \xi, \eta \rangle||\xi||\eta|.$$

We say that M satisfies $\mathrm{MTW(K)}$ if for any $\langle \xi . \eta \rangle = 0$

$$\mathfrak{S}_{(x,v)}(\xi, \eta) \geq K|\xi|^2|\eta|^2.$$

$\mathrm{MTW} \succeq 0$ stand for $\mathrm{MTW}(0)$ and $\mathrm{MTW} \succ 0$ for $\mathrm{MTW}(0)$ in a strict form.

Historic and definitions

Transport Continuity Property

Theorem (Figalli, Rifford, Villani '11)

The transport map T is continuous for any μ and ν two positive continuous measure, if

- *All the injectivity domains are strictly convex.*
- $\mathrm{MTW} \succ 0$.

Reciprocally if for any μ and ν two positive continuous measure the transport map T is continuous then

- *All the injectivity domains are convex.*
- $\mathrm{MTW} \succeq 0$.

In dimension 2, the transport map T is continuous for any μ and ν two positive continuous measure, if and only if

- *All the injectivity domains are convex.*
- $\mathrm{MTW} \succeq 0$.

Bonnet-Meyer's type theorem

Remark (Loeper)

Taking $x = y$, $\mathfrak{S}_{(x,0)}(\xi, \eta) = Sec(\xi, \eta)$

Theorem (Figalli, G., Rifford)

If M *is non focal then* $\mathrm{MTW} \succeq 0$ *implies that all the injectivity domains are convex.*

The extended tensor

let $x \in M$, $v \in \mathrm{NF}(x)$, and $(\xi, \eta) \in T_xM \times T_xM$.

$$\begin{array}{rcl} \Psi_{(x,v)} : \mathcal{V} \subset TM & \longrightarrow & \mathcal{W} \subset M \times M \\ (x', v') & \longmapsto & \left(x', \exp_{x'}(v')\right) \end{array}$$

is a smooth diffeomorphism from $\mathcal{V}$ to $\mathcal{W}$. Then we may define $\hat{c}_{(x,v)} : \mathcal{W} \to \mathbb{R}$ by

$$\hat{c}_{(x,v)}(x', y') := \frac{1}{2}\left|\Psi^{-1}_{(x,v)}(x', y')\right|^2_{x'}, \qquad \forall\, (x', y') \in \mathcal{W}.$$

The extended Ma–Trudinger–Wang tensor:

$$\overline{\mathfrak{S}}_{(x,v)}(\xi, \eta) = -\frac{3}{2}\; \frac{d^2}{ds^2}\bigg|_{s=0}\; \frac{d^2}{dt^2}\bigg|_{t=0}\; \hat{c}_{(x,v)}\Big(\exp_x(t\xi),\, \exp_x(v+s\eta)\Big).$$

Strategy

Proof: Kim-McCann's computation

Lemma

Let $x \in M$ and $v_0, v_1 \in \mathrm{I}(x)$. We define $h \in C^2([0,1], \mathbb{R}^+)$ by

$$h(t) = |\overline{q}_t|^2 - |q_t|^2 .$$

Then, for every $t \in [0,1]$ we have

$$\dot{h}(t) = \langle q_t - \overline{q}_t, \dot{y}_t \rangle_{y_t} ,$$

$$\ddot{h}(t) = \frac{2}{3} \int_0^1 (1-s)\, \overline{\mathfrak{S}}_{(y_t,(1-s)\overline{q}_t + sq_t)}(\dot{y}_t, q_t - \overline{q}_t)\, ds.$$

Proof: control lemma

Lemma

Let $h : [0,1] \to [0,\infty)$ be a C^2 function such that $h(0) = h(1) = 0$ and let $K, C > 0$ be two fixed constants. Assume that h satisfies:

$$\ddot{h}(t) \geq -C|\dot{h}(t)| - K \qquad \forall\, t \in [0,1]$$

Then

$$h(t) \leq 4Ke^{(1+C)}t(1-t) \qquad \forall\, t \in [0,1].$$

We conclude thanks to a bootstrap argument on K.

The control conditions

$\forall r > 0$ such that $B_x(r) \cap \mathrm{I}(x)$ is convex for all $x \in M$, there are $\overline{\beta}(r) > 0$ and a compact set $Z \subset TM$ with radial fibers satisfying the following properties

1. There are $C, D > 0$ such that $\overline{\mathrm{MTW}(-\mathrm{D}\rho, \mathrm{C})}$ holds on Z.
2. There is $K > 0$ such that

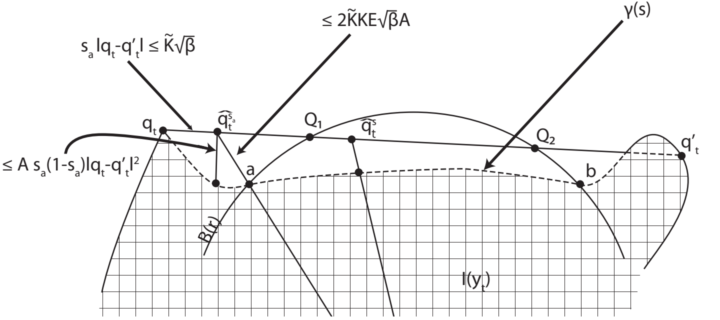
$$\rho_x\big(v, \mathrm{I}(x)\big) \leq K\left(|v|_x^2 - d\big(x, \exp_x(v)\big)^2\right) \qquad \forall (x,v) \in Z.$$

3. $\forall x \in M, \forall \beta \in (0, \overline{\beta}(r))$, $\mathrm{I}(x) \cap B_x(r+\beta) \subset Z(x) \subset \overline{\mathrm{NF}}(x)$.
4. $\forall x \in M, \forall \beta \in (0, \overline{\beta}(r)), \forall v_0, v_1 \in \mathrm{I}(x) \cap B_x(r+\beta)$, $v_t \in Z(x)$ and $[q_t, \overline{q}_t] \subset Z(y_t)$.

Then all injectivity domains of M are convex.

The non focal case

- Intersection with balls
- Initialisation with Lipschitz property
- Control properties are ok


≤ 2K̃KE√βA
γ(s)
s_a |q_t-q'_t| ≤ K̃√β
q_t
q̂_t^{s_a}
Q_1
q̂_t^s
Q_2
q'_t
≤ A s_a(1-s_a)|q_t-q'_t|²
a
b
B(r)
I(y_t)

Perspectives

- Get rid of the non-focal conditions.
- Understand the optimal transport near a purely focal speed.