Differential invariants of feedback transformations for quasi-harmonic oscillation equations

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Main aim

- Classification of quasi-harmonic oscillation equation with respect to feedback transformations.

- Problem formulation
- Admissible feedback transformations
- Differential invariants of quasi-harmonic oscillation equations
- Invariant differentiations of feedback transformations
- Algebras' of differential invariants dimensions
- Equivalence problem
- Canonical forms of equations

Problem formulation

Consider the following differential equation:

$$\frac{d^2y}{dx^2} + f(y,u) = 0, \qquad (1)$$

where the function $f(y,u)$ is smooth. Here u is a scalar control parameter. The problems of equivalence and classification for such an equation with respect to the feedback transformations:

$$\varphi : (x,y,u) \longmapsto (X(x,y), Y(x,y), U(x,y,u)) \qquad (2)$$

are solved.

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Denote a space of 2-jets of smooth functions by $J^2(\mathbb{R}^2)$. Then the canonical coordinates of this space are:

$$u, x, y, y_x, y_u, y_{xx}, y_{ux}, y_{uu}.$$

In the space considered equation (1) specifies a hyper-surface:

$$\mathcal{E}_f = \{y_{xx} + f(y, u) = 0\} \subset J^2(\mathbb{R}^2),$$

thereafter also called a control-parameter-dependent quasi-harmonic oscillaton equation (QHOE).

Let $\mathcal{E}_f$ and $\mathcal{E}_g$ be the equations of the form (1) corresponding to functions f and g accordingly. Suppose that

$$\varphi^{(2)}(\mathcal{E}_f) = \mathcal{E}_g, \tag{3}$$

for some function g, where the prolongation of feedback transformation φ to the space of 2-jets is denoted by $\varphi^{(2)}$. Then (3) is equivalent to:

$$\left(\varphi^{(2)}\right)(y_{xx} + f(y,u)) = \lambda\,(y_{xx} + g(y,u)),$$

where λ is a function at $J^2(\mathbb{R}^2)$.

Having followed an approach proposed by Sophus Lie, consider infinitesimal transformations:

$$\varphi_t : (x, y, u) \longmapsto (X_t(x, y), Y_t(x, y), U_t(u)) \qquad (4)$$

instead of generalized point transformations (2).

Here $X_t(x, y), Y_t(x, y), U_t(x, y, u)$ are the smooth functions of a parameter t.

At $t = 0$ φ_0 is an identical transformation. It means that:

$$X_0(x, y) = x, \quad Y_0(x, y) = y, \quad U_0(x, y, u) = u.$$

Determination of vector fields-1

Let us find the vector fields as follows:

$$X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y} + C(u)\frac{\partial}{\partial u}. \tag{5}$$

Thereafter the prolongations of vector fields and infinitesimal transformations in the space of 2-jets are denoted by $X^{(2)}$ and $\varphi_t^{(2)}$, respectively.

The class of equations operated by infinitesimal transformations is preserved under the following condition analogous to (3):

$$\left(\varphi_t^{(2)}\right)^* (y_{xx} + f(y,u)) = \lambda_t (y_{xx} + g_t(y,u)), \tag{6}$$

where λ_t is a local one parameter family of functions at $J^2(\mathbb{R}^2)$, and $g_t(y,u)$ is a local one parameter family of functions of variables y and u, such that:

$$\lambda_0 = 1, \quad g_0(y,u) = f(y,u). \tag{7}$$

Determination of vector fields-2

$$X = (\alpha x + \beta)\frac{\partial}{\partial x} + (\gamma + \frac{1}{2}\alpha y + \delta y)\frac{\partial}{\partial y} + C(u)\frac{\partial}{\partial u}. \quad (8)$$

Here $\alpha, \beta, \gamma, \delta$ are arbitrary constants and $C(u)$ is an arbitrary smooth function. Hence any vector fields preserving a class of equations (1) can be presented as a linear combination of the following basic vector fields with respect to feedback transformations:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = y\frac{\partial}{\partial y}, \quad X_4 = x\frac{\partial}{\partial x}, \quad X_5 = c(u)\frac{\partial}{\partial u}.$$

Local translation groups along vector fields

$$\varphi_{1,t} : (x, y, u) \longmapsto (x + t, y, u),$$
$$\varphi_{2,t} : (x, y, u) \longmapsto (x, y + t, u),$$
$$\varphi_{3,t} : (x, y, u) \longmapsto (x, e^t y, u),$$
$$\varphi_{4,t} : (x, y, u) \longmapsto (e^t x, y, u),$$
$$\varphi_{5,t} : (x, y, u) \longmapsto (x, y, U(u)).$$

Action of transformations on QHOE

Transformation $\varphi_{1,t}$ doesn't change the form of equation (1).

Applying transformations $\varphi^{(2)}_{2,t}$ - $\varphi^{(2)}_{5,t}$ to the left side of (1) result in:

$$\left(\varphi^{(2)}_{2,t}\right)^* (y_{xx} + f(y,u)) = y_{xx} + f(y+t,u)$$
$$\left(\varphi^{(2)}_{3,t}\right)^* (y_{xx} + f(y,u)) = y_{xx} + e^{-t} f(y,u),$$
$$\left(\varphi^{(2)}_{4,t}\right)^* (y_{xx} + f(y,u)) = y_{xx} + e^{\frac{1}{2}t} f(y,u),$$
$$\left(\varphi^{(2)}_{5,t}\right)^* (y_{xx} + f(y,u)) = y_{xx} + f(y,U(u)).$$

- Construct a trivial vector bundle with a base $\mathbb{R}^3$:

$$\pi : \mathbb{R}^3 \to \mathbb{R}, \qquad \pi : (y, u, z) \mapsto (y, u).$$

- The smooth functions

$$s_f : (y, u) \mapsto (y, u, f(y, u))$$

are the sections of this bundle.

Transformations $\varphi_{2,t}^{(2)}$ - $\varphi_{5,t}^{(2)}$ form a Lie pseudo-group.

Correspondent vector fields are the following:

$$Y_1 = \frac{\partial}{\partial y}, \quad Y_2 = y\frac{\partial}{\partial y}, \quad Y_3 = z\frac{\partial}{\partial z}, \quad Y_4 = H(u)\frac{\partial}{\partial u},$$

where H is an arbitrary smooth function.

These vector fields form a Lie algebra $\mathcal{G}$.

The differential invariants of equation (1) are the differential invariants of the Lie pseudo-group generated by a set of vector fields Y_1, Y_2, Y_3, Y_4.

Let $J^k(\pi)$ be the space of k-jets of a bundle π and $y, u, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}, \ldots$ be the canonical coordinates at this space. Denote the prolongations of vector fields $Y_i (i = 1, \ldots, 4)$ to the space $J^k(\pi)$ by $Y_i^{(k)}$.

Definition

The function J at the space of k-jets $J^k(\pi)$ smooth in its domain and rational with respect to variables z_σ on the fibers of bundle π is called *differential invariant of order* $\leq k$ of Lie (pseudo-)group G if it is a constant on the orbits of prolonged Lie (pseudo-)group $G^{(k)}$.

Having solved the system of differential equations:

$$Y^{(k)}(J) = 0, \tag{9}$$

we calculate differential invariants of order $\leq k$ of Lie pseudo-group G for any $Y \in \mathcal{G}$.

Basic second-order differential invariants

The system (9) being solved at $k = 2$ gives:

Theorem

Functions

$$J_{21} = \frac{z_{yy} z}{z_y^2}, \quad J_{22} = \frac{z_{yu} z}{z_y z_u}$$

form a complete set of the basic second-order differential invariants, i.e.any other second-order differential invariants are the functions of J_{21} *and* J_{22}*.*

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Definition

Operator

$$\nabla = M\frac{d}{dy} + N\frac{d}{du} \tag{10}$$

is called G-*invariant differentiation* if it commutes with every element of any prolongation of Lie algebra $\mathcal{G}$, where M and N are the functions on the jet space.

Invariant differentiations

Theorem

Differential operators

$$\nabla_1 = \frac{z}{z_y}\frac{d}{dy}, \tag{11}$$

$$\nabla_2 = \frac{z}{z_u}\frac{d}{du} \tag{12}$$

are G-invariant differentiations.

Here

$$\frac{d}{dy} = \frac{\partial}{\partial y} + z_x\frac{\partial}{\partial z} + z_{xx}\frac{\partial}{\partial z_x} + z_{xxx}\frac{\partial}{\partial z_{xx}} + \dots,$$
$$\frac{d}{du} = \frac{\partial}{\partial u} + z_u\frac{\partial}{\partial z} + z_{uu}\frac{\partial}{\partial z_u} + z_{uuu}\frac{\partial}{\partial z_{uu}} + \dots$$

are the operators of total differentiation with respect to the variables y and u.

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Let us remind that Lie algebra is generated by the following vector fields:

$$Y_1 = \frac{\partial}{\partial y}, \quad Y_2 = y\frac{\partial}{\partial y}, \quad Y_3 = z\frac{\partial}{\partial z}, \quad Y_4 = H(u)\frac{\partial}{\partial u}.$$

Theorem

Hyper-surface $z = 0$ divides the space $J^0(\pi)$ into two connected components. Lie pseudo-group G operates transitively at any connected component.

Theorem

Quasi-harmonic oscillation equation differential invariants' algebra is generated by second-order differential invariants J_{21}, J_{22} *and invariant differentiations* ∇_1 *and* ∇_2. *This algebra separates regular orbits. Algebra's of differential invariants of order* $\leq k$ *dimensions are equal to:*

$$\nu_k = \frac{k^2 + k - 2}{2}.$$

Regular equations

Definition

Let us call an equation $\mathcal{E}_f$ *regular*, if

$$dJ_{21}(f) \wedge dJ_{22}(f) \neq 0.$$

Here $J(f)$ — is the value of the differential invariant J on the function $f = f(y, u)$.

In a case of regular equations the coordinates y, u can be replaced by the functions $J_{21}(f)$, $J_{22}(f)$ on $\mathbb{R}^2$.

Since the functions $J_{31}(f)$, $J_{32}(f)$ and $J_{33}(f)$ are also the functions of y, u, there exists a functional dependence between the functions $J_{31}(f)$, $J_{32}(f)$ $J_{33}(f)$ and J_{21}, J_{22}:

$$J_{3i}(f) = \Phi_{fi}(J_{21}(f), J_{22}(f)).$$

Theorem

Suppose that the functions f and g are real-analytical. Two regular equations $\mathcal{E}_f$ and $\mathcal{E}_g$ are locally G-equivalent if and only if the functions Φ_{if} and Φ_{ig} identically equal ($i = 1, 2, 3$) and 3-jets of the functions f and g belong to the same connection component.

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Canonical forms

The following equations specify some of canonical forms resulted from equation of class (1).

Theorem

The equation (1) is locally equivalent to the following equation

$$\frac{d^2y}{dx^2} + \frac{b(u)^n}{n^n} y^n = 0$$

with respect to feedback transformations if and only if

$$J_{21}(f) = \frac{n-1}{n}, \quad J_{22}(f) = 1, \quad J_{31}(f) = \frac{n^2 - 3n + 2}{n^2},$$
$$J_{32}(f) = \frac{n-1}{n}, \quad J_{33}(f) = 0$$

for some natural number n.

Case n=1

$$J_{21} = 0, \quad J_{22} = 1, \quad J_{31} = 0, \quad J_{32} = 0, \quad J_{33} = 0.$$

Theorem

The equation (1) is locally equivalent to the following equation

$$\frac{d^2y}{dx^2} + \alpha(u)y + \beta(u) = 0$$

if and only if $J_{21}(f) = 0$.

Theorem

The equation (1) is locally equivalent to the following equation

$$\frac{d^2y}{dx^2} + \alpha(u)y^2 + \beta(u)y + \gamma(u) = 0$$

if and only if $J_{31}(f) = 0$.

Results

- The classification problem for a control-parameter-dependent quasi-harmonic oscillation equation with respect to feedback transformations has been solved.
- The structure of the algebra of differential invariants has been described.
- The equivalence problem has been solved.
- Some canonical forms have been specified.

THANK YOU

FOR ATTENTION!!!

Taking the derivative of (6) with respect to t at $t = 0$ and accounting for (7), we obtain:

$$\frac{d}{dt}\bigg|_{t=0} \left(\varphi_t^{(2)}\right)^* (y_{xx} + f(y,u)) = \lambda_0 \frac{d}{dt}\bigg|_{t=0} (y_{xx} + g_t(y,u)) + \tag{13}$$

$$+ (y_{xx} + g_0(y,u)) \frac{d\lambda_t}{dt}\bigg|_{t=0} = \frac{d}{dt}\bigg|_{t=0} g_t(y,u) + (y_{xx} + f(y,u)) \frac{d\lambda_t}{dt}\bigg|_{t=0}.$$

The left side of (13) is a Lie derivative along the vector field $X^{(2)}$ of a function $y_{xx} + f(y_0, u)$. A restriction of the last equality of (13) on $\mathcal{E}_f$ is:

$$L_{X^{(2)}}(y_{xx} + f(y,u))|_{\mathcal{E}_f} = \lambda_0 G(y,u), \tag{14}$$

where

$$G(y,u) = \frac{d}{dt}\bigg|_{t=0} (y_{xx} + g_t(y,u))\bigg|_{\mathcal{E}_f}.$$

The prolongations of the vector fields Y_i to the space of 2-jets

$$
\begin{aligned}
Y_1^{(2)} &= Y_1, \\
Y_2^{(2)} &= Y_2 - z_y \frac{\partial}{\partial z_y} - 2 z_{yy} \frac{\partial}{\partial z_{yy}} - 2 z_{yu} \frac{\partial}{\partial z_{yu}}, \\
Y_3^{(2)} &= Y_3 + z_y \frac{\partial}{\partial z_y} + z_u \frac{\partial}{\partial z_u} + z_{yy} \frac{\partial}{\partial z_{yy}} + z_{yu} \frac{\partial}{\partial z_{yu}} + z_{uu} \frac{\partial}{\partial z_{uu}}, \\
Y_4^{(2)} &= Y_4 - H_u(u) z_u \frac{\partial}{\partial z_u} - H_u(u) z_{yu} \frac{\partial}{\partial z_{yu}} - \Big(H_{uu}(u) z_u + 2 H_u(u) z_{uu} \Big) \frac{\partial}{\partial z_{uu}}.
\end{aligned}
$$

Basic third-order differential invariants

Resolving system (9) at $k = 3$ we obtain

Theorem

Functions

$$J_{31} = \frac{z_{yyy} z^2}{z_y^3}, \quad J_{32} = \frac{z_{yyu} z^2}{z_y^2 z_u}, \quad J_{33} = \frac{z_{yuu} z^2}{z_u^2 z_y} - J_{22} \frac{z_{uu} z}{z_u^2}$$

form a complete set of the basic third-order differential invariants.

Basic fourth-order differential invariants

Theorem

Functions

$$J_{41} = \frac{z_{yyyy} z^3}{z_y^4}, \quad J_{42} = \frac{z_{yyyu} z^3}{z_y^3 z_u},$$

$$J_{43} = \frac{z_{yyuu} z^3}{z_u^2 z_y^2} - J_{32} \frac{z_{uu} z}{z_u^2} \quad J_{44} = \quad \frac{z_{yuuu} z^3}{z_y z_u^3} - 3J_{33} \frac{z_{uu} z}{z_u^2} - J_{22} \frac{z_{uuu} z^2}{z_u^3}$$

form a complete set of the basic fourth-order differential invariants.

Basic fifth-order differential invariants

Theorem

Functions

$$J_{51} = \frac{z_{yyyyy} z^4}{z_y^5}, \quad J_{52} = \frac{z_{yyyyu} z^4}{z_y^4 z_u}, \quad J_{53} = \frac{z_{yyyuu} z^4}{z_u^3 z_y^2} - J_{42} \frac{z_{uu} z}{z_u^2},$$

$$J_{54} = \frac{z_{yyuuu} z^4}{z_y^2 z_u^3} - 3J_{43} \frac{z_{uu} z}{z_u^2} - J_{32} \frac{z_{uuu} z^2}{z_u^3},$$

$$J_{55} = \frac{z_{yuuuu} z^4}{z_y z_u^4} - 3J_{44} \frac{z_{uu} z}{z_u^2} - J_{33} \left(\frac{z_{uu} z}{z_u^2} \right)^2 - J_{33} \frac{z_{uuu} z^2}{z_u^3} - J_{22} \frac{z_{uuuu} z^3}{z_u^4}$$

form a complete set of the basic fifth-order differential invariants.

Action of invariant differentiations-1

$$
\begin{aligned}
\nabla_1(J_{21}) &= J_{21} - 2J_{21}^2 + J_{31},\\
\nabla_2(J_{21}) &= J_{21} - 2J_{21}J_{22} + J_{32},\\
\nabla_1(J_{22}) &= J_{22} - J_{21}J_{22} - J_{22}^2 + J_{32},\\
\nabla_2(J_{22}) &= J_{22} - J_{22}^2 + J_{33}.
\end{aligned}
$$

$$
\begin{aligned}
\nabla_1(J_{31}) &= 2J_{31} - 3J_{21}J_{31} + J_{41},\\
\nabla_2(J_{31}) &= 2J_{31} - 3J_{22}J_{31} + J_{42},\\
\nabla_1(J_{32}) &= 2J_{32} - 2J_{21}J_{32} - J_{22}J_{32} + J_{42},\\
\nabla_2(J_{32}) &= 2J_{32} - 2J_{22}J_{32} + J_{43},\\
\nabla_1(J_{33}) &= 2J_{33} - 2J_{21}J_{33} - 3J_{22}J_{33} + J_{43},\\
\nabla_2(J_{33}) &= 2J_{33} - J_{22}J_{33} + J_{44}.
\end{aligned}
$$

Action of invariant differentiations-2

$$\nabla_1(J_{41}) = 3J_{41} - 4J_{21}J_{41} + J_{51},$$
$$\nabla_2(J_{41}) = 3J_{41} - 4J_{22}J_{41} + J_{52},$$
$$\nabla_1(J_{42}) = 3J_{42} - 3J_{21}J_{42} - J_{22}J_{42} + J_{52},$$
$$\nabla_2(J_{42}) = 3J_{42} - 3J_{22}J_{42} + J_{53},$$
$$\nabla_1(J_{43}) = 3J_{43} - 2J_{21}J_{43} - 2J_{22}J_{43} - J_{33}J_{32} + J_{53},$$
$$\nabla_2(J_{43}) = 3J_{43} - 2J_{22}J_{43} + J_{54},$$
$$\nabla_1(J_{44}) = 3J_{44} - J_{21}J_{44} - 4J_{22}J_{44} - J_{33}^2 + J_{54},$$
$$\nabla_2(J_{44}) = 3J_{44} - J_{22}J_{44} + J_{55}.$$