ABSENCE OF LOCAL MAXIMA FOR OPTIMAL CONTROL OF TWO-LEVEL QUANTUM SYSTEMS

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Pechen A. N. and Il'in N. B. Trap-free manipulation in the Landau-Zener system. *Phys. Rev. A* **86**, 052117 (2012).
Pechen, Il'in. *Proceedings of the Steklov Institute of Mathematics.* (2014) in press.

CONTENTS

1. Our main result: no traps in the two-level system

Theorem: Consider two-level quantum system with controlled evolution

$$i\frac{d}{dt}U_t^f = [H_0 + f(t)V]U_t^f$$

$$H_0, V \in u(2, \mathbb{C}), f(t) \in L^1[0, T]$$

If $\mathrm{Tr}V = 0$ and $T \geqslant \pi/(|H_0 - \frac{1}{2}\mathrm{Tr}H_0 + V|)$, $f_0 = -\mathrm{Tr}(H_0V)/\mathrm{Tr}(V^2)$, then all maxima of functionals $J_{i\to f}(f) = |\langle\psi_f|U_T^f|\psi_i\rangle|^2$, $J_O(f) = \langle\psi_f|U_T^{f\dagger}OU_T^f|\psi_i\rangle$, $J_W(f) = |\mathrm{Tr}(U_T^fW^\dagger)|^2$ (and much more general) are global that is, there are no local maxima.

2. Formulation of a general quantum control problem

Formulation:

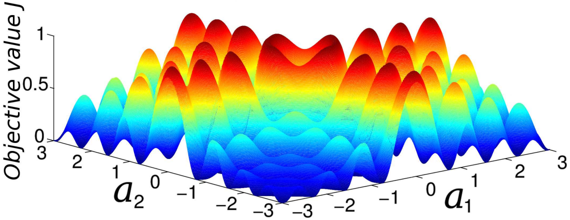
- Unitary evolution: $i\dot{U}_t^f = [H_0 + f(t)V]U_t^f$
- Control: $f \in L^1[0, T]$
- The goal is to maximize objective $J(f) = F(U_T^f)$, $F : U(n) \to \mathbb{R}$
  - Transition probability $J_{i\to f}(f) = |\langle\psi_f|U_T^f|\psi_i\rangle|^2$
  - Average value $J_O(f) = \langle\psi_f|U_T^{f\dagger} O U_T^f|\psi_i\rangle$
  - Gate generation $J_W(f) = |\mathrm{Tr}(U_T^f W^\dagger)|^2$

Brief history:

- 1980-th: First works on quantum control.
- Now: Quantum control is an actively growing field. More than 1300 articles published every year according to *WebOfScience*.

3. Traps of $J(f)$

Trap: local maximum of $J(f)$ with objective value less than the global maximum.



Importance: The existence of traps may prevent local algorithms from finding the global maximum. This circumstances motivates hight interest to the analysis of traps.

4. Overview of previous results about traps

- Rabitz, Hsieh and Rosenthal (Science 2004): Conjecture of the absence of traps for quantum systems (not proved!).[1]
- Pechen and Tannor (PRL 2012): Prove the existence of some kind of traps for $n > 2$-level quantum systems with special symmetries[2] and hence "show mathematically that the situation is more complicated".[3]
- Fouquieres and Schirmer (IDAQP 2013):[4] Constructed examples of traps.
- Pechen and Il'in (PRA 2012)and subsequent work (PSIM 2014)[5]: Prove the absence of traps for the two-level system. Our work provides a first example of a trap-free quantum control system. Prior this work no examples of trap-free systems were known.

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[1] Rabitz, Hsieh, Rosenthal. *Science* **303**, 1998 (2004).
[2] Pechen, Tannor. *Phys. Rev. Lett.* **108**, 198902 (2012).
[3] Yeston. *Science* **332**, 514 (2011).
[4] Fouquieres, Schirmer. *IDAQP* **16(3)**, 1350021 (2013).
[5] Pechen, Il'in. *Proceedings of the Steklov Institute of Mathematics.* (2014)

5. Details of our main result

Two-level system: $i\dot{U}_t^f = [H_0 + f(t)V]U_t^f$

Importance: Two-level system models a wide range of phenomena in physics, chemistry and quantum information.

Theorem:(main result) There are no traps for objective $J_O(f)$, $J_{i\to f}(f)$, $J_W(f)$ (and for more general objectives) for the two-level system.

Proof (Sketch):

- $\nabla_f J(U_T^f) = 0 \Rightarrow \nabla_U J(U) \circ \nabla_f U_T^f = 0$
- $\nabla_f U_T^f = -iU_T^f U_t^{f\dagger} V U_t^f$
- $\nabla_U J(U)\Big|_{U=\nabla_f U_T^f} = 0 \Rightarrow \nabla_U J[-iU_T^f X] = 0, \forall X \in su(2)$

5. Details of our main result

Let $L(X) := \nabla_U J[-iU_T^f X]$, $V_t := U_t^{f\dagger} V U_t^f$. Consider the function $l(t) := L(V_t)$. The equality $L(V_t) = 0$ means $l(t) \equiv 0$. Therefore, in particular, $l(t) = l'(t) = l''(t) = 0$, that implies

- $0 = L(U_t^{f\dagger} V U_t)$
- $0 = L(U_t^{f\dagger}[H_0, V]U_t)$
- $0 = L(U_t^{f\dagger}([H_0,[H_0,V]] + f(t)[V,[H_0,V]])U_t)$

If $f(t)$ is not equal identically to f_0 then there exists t such that the matrices $\mathbb{I}$, V, $[H_0, V]$, and $E_t = [H_0,[H_0,V]] + f(t)[V,[H_0,V]]$ are linearly independent.

This finishes outline of the proof.

Remark: Our proof assumes ideal case with unconstrained controls and without noise.

6. Deviations from the ideal case

In order to estimate deviations from the ideal case, we consider the system of special form

$$i\frac{d}{dt}U_t^f = [\sigma_x + f(t)\sigma_z]U_t^f$$

σ_x and σ_z are Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

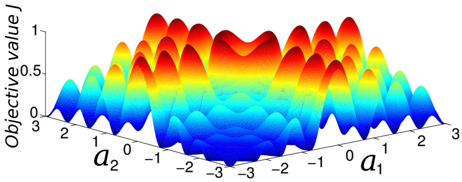
$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k,\ i, j, k = x, y, z$

6. Deviations from the ideal case: constrained controls

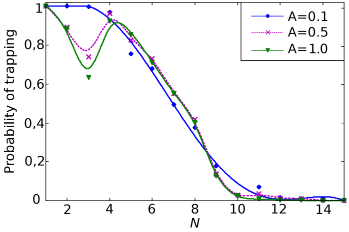
Control field in experiments often has the following form

$$\tilde{f}(t) = \sum_{i=1}^{N} a_i \chi_{[t_{i+1}, t_i]}(t), \quad a_i \in [-A, A]$$

Our result: Constrained objective $J(\tilde{f}) = \tilde{J}(a_1, \dots, a_n)$ has traps for small N.



Control landscape and traps for $N = 2$



Probability of trapping for different N

7. Deviations from the ideal case: noise in the control

Often in experiment control has noise component described by white noise with correlation function $\langle \xi(t)\xi(t')\rangle = \sigma^2\delta(t-t')$

- Additive: $f \to f + \xi(t)$
- Multiplicative: $f \to f(1+\xi(t))$

Our result: Noise decreases $\max J(f)$

$$\max\langle J(f)\rangle = \max J(f) - \sigma^2 D + o(\sigma^2)$$

Additive noise: $D_{AWN} \leq T$ (T — control pulse duration)

Multiplicative noise: $D_{MWN} \leq E$ ($E = \int_0^T |f_{max}(t)|^2 dt$ — energy of the optimal control f_{max})

Conclusions

In this talk we presented our theorem for the absence of traps in the two-level quantum system, discuss the limitations associated with the noise and constrains in the controls and provide an overview of previous results for the analysis of traps.

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Thank you!