Diffusion by optimal transport in the Heisenberg group

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Nicolas JUILLET

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The geometry of $\mathbb{H}$.

- A basis for left-invariant vector fields is

$$\mathbf{X} = E_1 - \frac{y}{2}E_3 \quad , \quad \mathbf{Y} = E_2 + \frac{x}{2}E_3 \quad \text{and} \quad \mathbf{Z} = [\mathbf{X}, \mathbf{Y}] = E_3.$$

- A curve is horizontal if $\dot{\gamma} \in \mathrm{Vect}(\mathbf{X}, \mathbf{Y})$ for any t. Actually

$$\dot{\gamma}(t) = a(t)\mathbf{X}(\gamma(t)) + b(t)\mathbf{Y}(\gamma(t)).$$

  It has norm $|\dot{\gamma}| = \sqrt{a^2(t) + b^2(t)}$.

- The diffusion operator is $\Delta = \mathbf{X}^2 + \mathbf{Y}^2$.

The Riemannian Heisenberg group $\mathbb{H}_\varepsilon$

Let $\varepsilon > 0$ and $\mathbb{H}_\varepsilon := (\mathbb{R}^3, d_\varepsilon, \mathcal{L}^3)$.

- An orthonormal basis at point (x, y, u) is

$$\mathbf{X} = E_1 - \frac{y}{2}E_3 \quad , \quad \mathbf{Y} = E_2 + \frac{x}{2}E_3 \quad \text{and} \quad \varepsilon\mathbf{Z} = \varepsilon[\mathbf{X}, \mathbf{Y}] = \varepsilon E_3.$$

- If

$$\dot{\gamma}(t) = (a(t)\mathbf{X} + b(t)\mathbf{Y} + c(t)\mathbf{Z})(\gamma(t))$$

  then $|\dot{\gamma}(t)|_\varepsilon = \sqrt{a^2 + b^2 + \frac{c^2}{\varepsilon^2}}$.

- The gradient is $\nabla_\varepsilon f = \nabla f + (\varepsilon^2 \mathbf{Z} f)\mathbf{Z}$.

Wasserstein metrics

Let W be the L^2-minimal metric with respect to the Carnot-Carathéodory metric d_{CC} and W_ε with respect to d_ε.

- $W_\varepsilon \leq W$,
- $\mathcal{P}_2(\mathbb{H}) = \mathcal{P}_2(\mathbb{H}_\varepsilon)$ as topological spaces.
- Lipschitz curves (resp. absolutely continuous) of $\mathcal{P}_2(\mathbb{H})$ are Lipschitz (resp. absolutely continuous) in $\mathcal{P}_2(\mathbb{H}_\varepsilon)$.

The relative entropy

The relative entropy of $\mu = \rho\mathcal{L}$ is given by

$$H(\mu) = H(\mu \mid \mathcal{L}) = \int \rho \ln(\rho)(x) d\mathcal{L}(x).$$

Big entropy: μ concentrated on a few space.
Small entropy: μ take a lot of space.

K-convexity and Ricci curvature

Theorem (Cordero–McCann–Schmuckenschläger and Sturm–von Renesse)

Let M be a Riemannian manifold.
It has Ricci curvature bounded from below by $K \in \mathbb{R}$ if and only if for every geodesic $(\mu_t)_{t\in[0,T]}$ in $\mathcal{P}_2(M)$ of speed 1 the function

$$t \in [0,T] \mapsto H(\mu_t \mid \mathrm{Vol}_M) - K.\frac{t^2}{2}$$

is convex.

No lower curvature bound for the Heisenberg group.

Theorem (J.)

In $\mathcal{P}_2(\mathbb{H})$ the entropy H with respect to $\mathcal{L}$ is not K-convex along geodesics (for any $K \in \mathbb{R}$).

The curvature-dimension conditons introduced by Lott–Villani and Sturm don't hold

Gradient flow of the relative entropy.

An absolutely continuous curve $(\mu_t)_{t\geq 0}$ is a gradient flow of H if

- $t \mapsto H(\mu_t)$ is decreasing.
- $$\text{for almost every } t>0, \begin{cases} \partial_t H(\mu_t) = -\text{Speed}(\mu_t)\cdot\text{Slope}(H)(\mu_t) \\ \text{Speed}(\mu_t) = \text{Slope}(H)(\mu_t). \end{cases}$$

Theorem (J.)

The curve $(\mu_t)_{t>0}$ is a gradient flow of H if and only if $\mu_t = \rho_t \mathcal{L}^3$ with

$$\frac{\partial}{\partial t}\rho_t = \Delta \rho_t.$$

HWI inequalities

From

$$H(\mu) \geq H(\nu) - \sqrt{I_\varepsilon(\nu)}W_\varepsilon(\mu,\nu) + \frac{1}{2} \cdot \left(\frac{-1}{2\varepsilon^2}\right) W_\varepsilon(\mu,\nu)^2$$

to

$$H(\mu) \geq H(\nu) - \sqrt{I(\nu)}W(\mu,\nu) - C(\mu)W(\mu,\nu)^{3/2},$$

but only if $I_\varepsilon(\nu) = I(\nu) + \varepsilon^2 J(\nu)$ is finite.

$\mathrm{Slope}_\varepsilon = \sqrt{I + \varepsilon^2 J}$ is a strong upper gradient of H on $\mathcal{P}_2(\mathbb{H}_\varepsilon)$

Let $(\mu_t)_{t\geq 0}$ be an absolutely continuous curve of $\mathcal{P}_2(\mathbb{H}_\varepsilon)$.

If

$$\int_{t_0}^{t_1} \mathrm{Slope}_\varepsilon(H)(\mu_t)\,\mathrm{Speed}_\varepsilon(\mu_t) < +\infty$$

then $t \mapsto H(\mu_t)$ is absolutely continuous and

$$|H(\mu_{t_0}) - H(\mu_{t_1})| \leq \int_{t_0}^{t_1} \mathrm{Slope}_\varepsilon(H)(\mu_t)\,\mathrm{Speed}_\varepsilon(\mu_t)\mathrm{d}t$$

We can prove that $\sqrt{I}(\neq \mathrm{Slope})$ is a strong upper gradient of $\mathcal{P}_2(\mathbb{H})$.