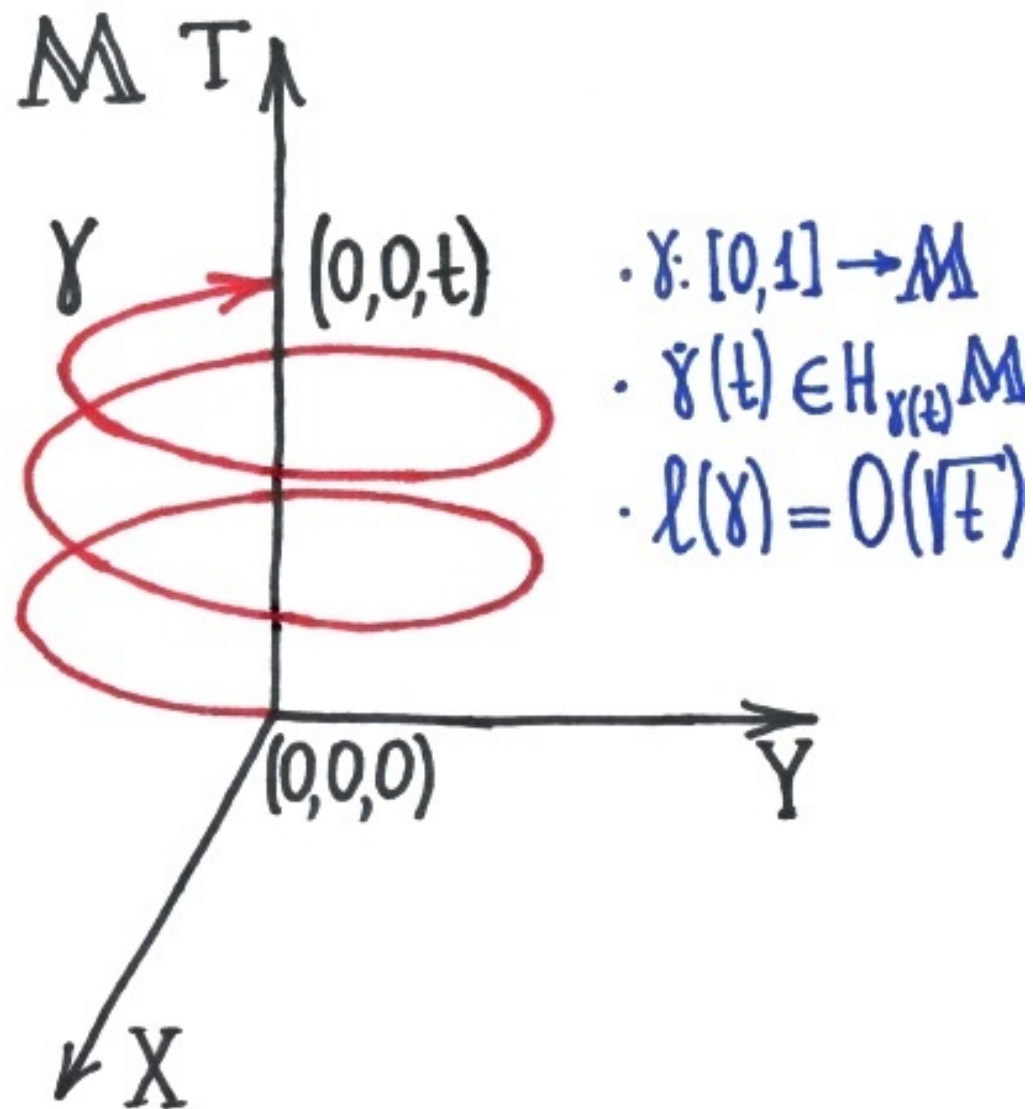
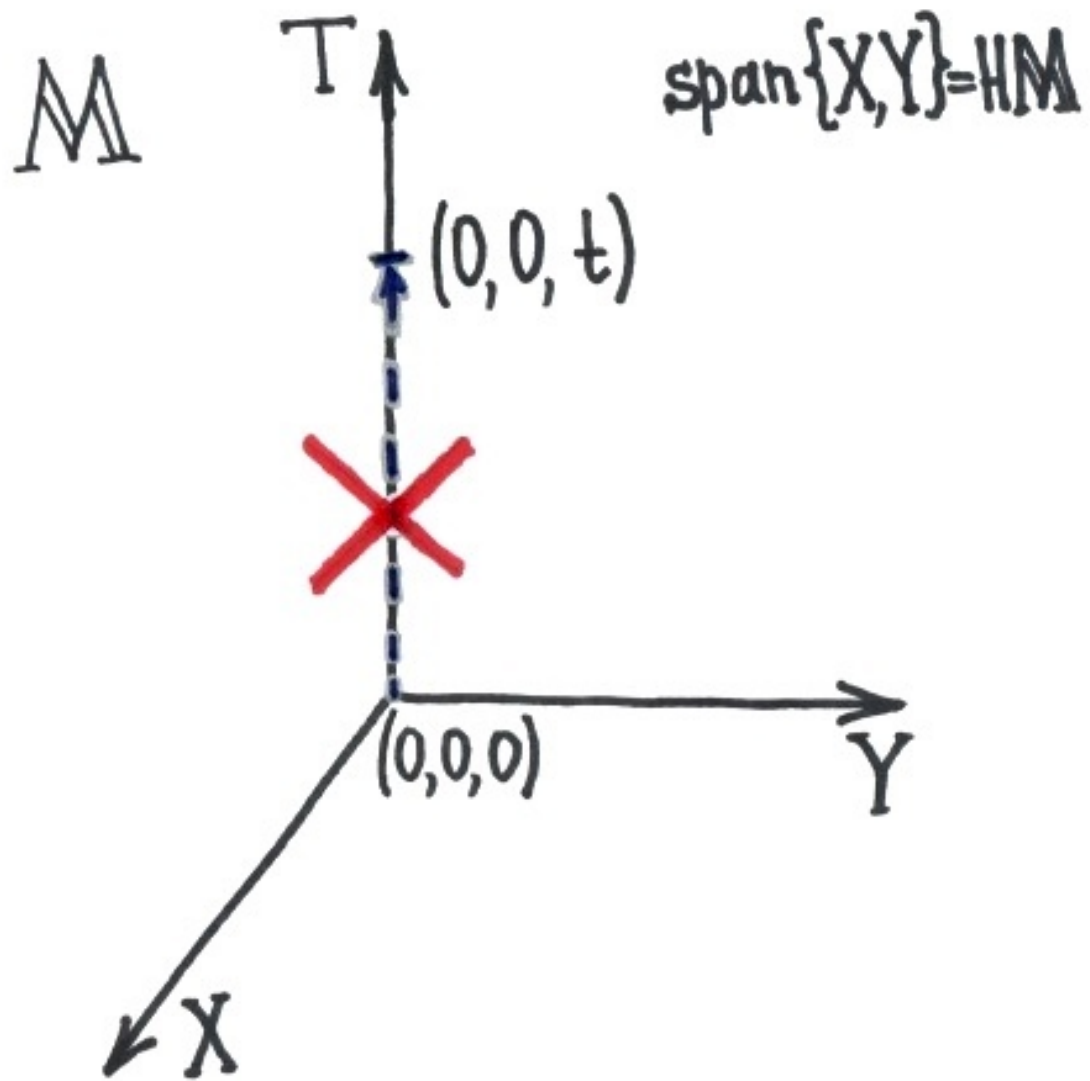


Metric Geometry of Carnot–Carathéodory Spaces and Its Applications

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Main Results and Their Direct Corollaries

- ◇ new metric properties of Carnot–Carathéodory spaces
- ◇ extension from classical results to the minimal smoothness
- ⇒ sub-Riemannian differentiability theory (Vodopyanov)
- ⇒ non-equiregular Carnot–Carathéodory spaces (Selivanova)
- ⇒ problem on attainable set (C^1 -fields) (Basalaev, Vodopyanov)
- ⇒ coarea formula for smooth contact mapping of CC-spaces
- ⇒ area formula for intrinsic Lipschitz mappings of CC-spaces
- ⇒ minimal surfaces-«graphs» on classes of Carnot groups

Carnot–Carathéodory Space (Smooth Vector Fields)

- M is a connected Riemannian C^∞ -smooth manifold
- $\dim_{\text{top}}(M) = N$
- $\exists H = HM \subset TM$, HM that generates the tangent bundle TM :
$$H = H_1 \subsetneq \text{span}\{H_1, [H_1, H_1]\} =: H_2$$
$$\subsetneq \text{span}\{H_2, [H_1, H_2]\} =: H_3 \subsetneq \dots \subsetneq H_M = TM$$
- $\dim H_i$'s do not depend on x (*equiregular* case; differs from Hörmander's one)

Hypoelliptic Equations

A problem: when a distribution solution f to the equation

$$(X_1^2 + \dots + X_{n-1}^2 - X_n)f = \varphi \in C^\infty$$

is a smooth function?

- **Particular case: Kolmogorov's equations**

$$\frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} = f$$

- physics (diffusion process), economics (arbitrage theory, some stochastic volatility models of European options), etc.

Hypoelliptic Equations

- Hörmander (1967): **sufficient conditions** on fields X_1, \dots, X_n :

Let $H_1(u) = \text{span}\{X_1(u), \dots, X_n(u)\}$. There exists $M < \infty$ s. t.

$$H_1(u) \subsetneq \text{span}\{H_1(u), [H_1, H_1](u)\} =: H_2(u)$$

$$\subsetneq \text{span}\{H_2(u), [H_1, H_2](u)\} =: H_3(u) \subsetneq \dots \subsetneq H_M(u) = T_u\mathbb{R}^N$$

for all points $u \in D$.

- Stein (1971): The program of studying of geometry of Hörmander vector fields;
description of singularities of fundamental solutions

Main Classical Results

- 1970 Goodman R. W.
- 1976 Rothschild L., Stein E.
- 1976 Metivier G.
- 1985 Nagel A., Stein E. M., Wainger S.
- 1985 Mitchell J.
- 1994 Berestovskii V.N.
- 1994 A. M. Vershik and V. Ya. Gershkovich
- 1994 M. I. Zelikin and V. F. Borisov
- 1996 Gromov M.
- 1996 Belläiche A.
- 2000 Margulis G. A., Mostow G. D.
- 2001 Agrachev A. A., Gauthier J.-P.
- 2002 Montgomery R.
- 2003 Agrachev A., Marigo A.
- 2008 J. Petitot

Applications

- subelliptic equations
 - non-holonomic mechanics
 - contact geometry
 - physics
 - robotics (C^1 -fields are important)
 - neurobiology (C^1 -fields are important)
- etc.

A Carnot–Carathéodory Space (C^1 -Fields)

- \mathbb{M} is a connected Riemannian C^∞ -manifold, $\dim_{\text{top}}(\mathbb{M}) = N$
- in $T\mathbb{M}$, there is a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \dots \subsetneq H_i\mathbb{M} \subsetneq \dots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

- $\forall v \in \mathbb{M} \exists U(v)$ with C^1 -smooth vector fields X_1, X_2, \dots, X_N , s. t.
- $H_i\mathbb{M}(v) = \text{span}\{X_1(v), \dots, X_{\dim H_i}(v)\}$, $\dim H_i\mathbb{M}(v) = \dim H_i$;
- (*) $[H_i, H_j] \subseteq H_{i+j}$, $i, j = 1, \dots, M$.

If $H_{j+1} = \text{span}\{H_j, [H_1, H_j], [H_2, H_{j-1}], \dots, [H_k, H_{j+1-k}]\}$, where $k = \lfloor \frac{j+1}{2} \rfloor$, $j = 1, \dots, M-1$ **then** \mathbb{M} is a **Carnot manifold**.

- ◇ $\deg X_i = \min\{k : X_i \in H_k\}$, $i = 1, \dots, N$

An Example of a Carnot–Carathéodory Space with C^1 -Smooth Basis Vector Fields (Answer to Gromov’s Question)

- $\psi, \varphi, \xi, \eta, \omega \in C^1([a, b], \mathbb{R}) \setminus C^2$; $\psi, \varphi, \xi, \eta, \omega \neq 0$, $\frac{d\varphi}{dy} \neq 0$ on $[a, b]$

$$X = \psi(x)\partial_x + \psi(x)\varphi(y)\partial_y + \eta\left(-\int_a^y \tilde{\xi}(x, s) ds + z\right)\partial_z + \varphi(y)\omega(q)\partial_q,$$

$$Y = \partial_y + \xi\left(-\int_a^y \frac{dt}{\varphi(t)} + x\right)\partial_z,$$

$$Z = -\psi(x)\partial_y + \omega(q)\partial_q,$$

$$T = \partial_y. \quad \text{Here } \tilde{\xi}(x, s) = \xi\left(-\int_a^s \frac{dt}{\varphi(t)} + x\right).$$

$$\Rightarrow X \in C^1(x, y, z, q), Y \in C^1(x, y), Z \in C^1(x, q), T \in C^\infty$$

- $[X, Y] = \frac{d\varphi}{dy} \cdot Z$ and $[Y, Z] = -\frac{d\psi}{dx} \cdot T$
- $H = H_1 = \text{span}\{X, Y\}$, $H_2 = \text{span}\{X, Y, Z\}$, $H_3 = \text{span}\{X, Y, Z, T\}$

Peculiarities and Difficulties (C^1 -Smooth Vector Fields)

? What structure approximates CC-space and how?

▶ Gromov's Theorem or Baker–Campbell–Hausdorff formula...

⊖ They are unknown!

? What (quasi)metric should we use?

▶ Rashevsky–Chow Theorem (d_{cc}) / generalized Δ inequality (d_∞)...

⊖ They are unknown!

A Local Lie Group at $u \in \mathbb{M}$ (C^1 -Smooth Vector Fields)

$$(*) \Rightarrow [X_i, X_j](v) = \sum_{k: \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v),$$

Property. Coefficients $\{c_{ijk}(u)\}_{\deg X_k = \deg X_i + \deg X_j} = \{\bar{c}_{ijk}\}$ enjoy Jacobi identity:

$$\sum_k \bar{c}_{ijk}(u) \bar{c}_{kml}(u) + \sum_k \bar{c}_{mik}(u) \bar{c}_{kjl}(u) + \sum_k \bar{c}_{jmk}(u) \bar{c}_{kil}(u) = 0$$

for all $i, j, m, l = 1, \dots, N$, and $\bar{c}_{ijk} = -\bar{c}_{jik}$ for all $i, j, k = 1, \dots, N$
 \Rightarrow nilpotent Lie algebra

basis vector fields $\{(\widehat{X}_i^u)'\}_{i=1}^N$ in \mathbb{R}^N are chosen in a such a way that $\exp = \text{Id}$. This implies $(\widehat{X}_i^u)'(0) = e_i, i = 1, \dots, N$

\Rightarrow corresponding Lie group: nilpotent Lie group

Local Homogeneous Lie Group $\mathcal{G}^u \subset \mathbb{M}$

$\mathcal{G} \subset \mathbb{M}$: a neighborhood of u with basis fields $\widehat{X}_i^u = D\theta_u(\widehat{X}_i^u)'$

$\theta_u : U(0) \rightarrow \mathcal{G}$ group isomorphism \Rightarrow structure of local Lie group

For $x = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right)(u)$, $y = \exp\left(\sum_{i=1}^N y_i \widehat{X}_i^u\right)(u)$, define

$$x \cdot y = \exp\left(\sum_{i=1}^N z_i \widehat{X}_i^u\right)(u),$$

where

$$z_i = x_i + y_i + \sum_{|\mu+\beta|_h = \deg X_i, \mu, \beta > 0} F_{\mu, \beta}^i(u) x^\mu y^\beta.$$

- fields $\widehat{X}_i^u = D\theta_u(\widehat{X}_i^u)'$ are left-invariant

Distance Functions

- If $w = \exp\left(\sum_{i=1}^N w_i \widehat{X}_i^u\right)(v)$ then $d_\infty^u(v, w) = \max_{i=1, \dots, N} \{|w_i|^{\frac{1}{\deg X_i}}\}$

- d_∞^u is a quasimetric: $\exists C = C(\mathcal{U})$ such that

$$d_\infty^u(v, w) \leq C(d_\infty^u(v, z) + d_\infty^u(z, w)) \quad \forall w, v, z \in \mathcal{U}$$

- If $v = \exp\left(\sum_{i=1}^N v_i \widehat{X}_i^u\right)(u)$ then $\Delta_\varepsilon^u(v) = \exp\left(\sum_{i=1}^N \varepsilon^{\deg X_i} v_i \widehat{X}_i^u\right)(u)$

- If $w = \exp\left(\sum_{i=1}^N w_i X_i\right)(v)$ then $d_\infty^u(v, w) = \max_{i=1, \dots, N} \{|w_i|^{\frac{1}{\deg X_i}}\}$

- Main Result $\Rightarrow d_\infty$ is a quasimetric

Main Result

- \mathbb{M} : a CC-space with $C^{1,\alpha}$ -smooth basis vector fields
- $\alpha \in [0, 1]$; $\alpha = 0 \Rightarrow$ the vector fields are just C^1 -smooth

Theorem. $\forall w \in \mathbb{M} \exists \mathcal{O} \ni w, \mathcal{O} \in \mathbb{M}$, such that $\forall u, x \in \mathcal{O}$ and $\forall q = 1, \dots, N$ the representation holds

$$X_q(\Delta_\varepsilon^u x) = \sum_{p=1}^N a_{p,q}^u(\Delta_\varepsilon^u x) \widehat{X}_p^u(\Delta_\varepsilon^u x),$$

where

$$a_{p,q}^u(\Delta_\varepsilon^u x) = \begin{cases} O(\varepsilon), & \deg X_p < \deg X_q, \\ \delta_{pq} + O(\varepsilon), & \deg X_p = \deg X_q, \\ O(\varepsilon^{\alpha + \deg X_p - \deg X_q}), & \deg X_p > \deg X_q, \quad \alpha > 0, \\ o(\varepsilon^{\deg X_p - \deg X_q}), & \deg X_p > \deg X_q, \quad \alpha = 0. \end{cases}$$

Comparison of Local Geometries (Local Groups)

- $\mathcal{U} \in \mathbb{M}$ is an arbitrary neighborhood described in the main result
- $d_\infty(w, v)$ and $d_\infty^u(w, v)$ are well-defined $\forall u, v, w \in \mathcal{U}$

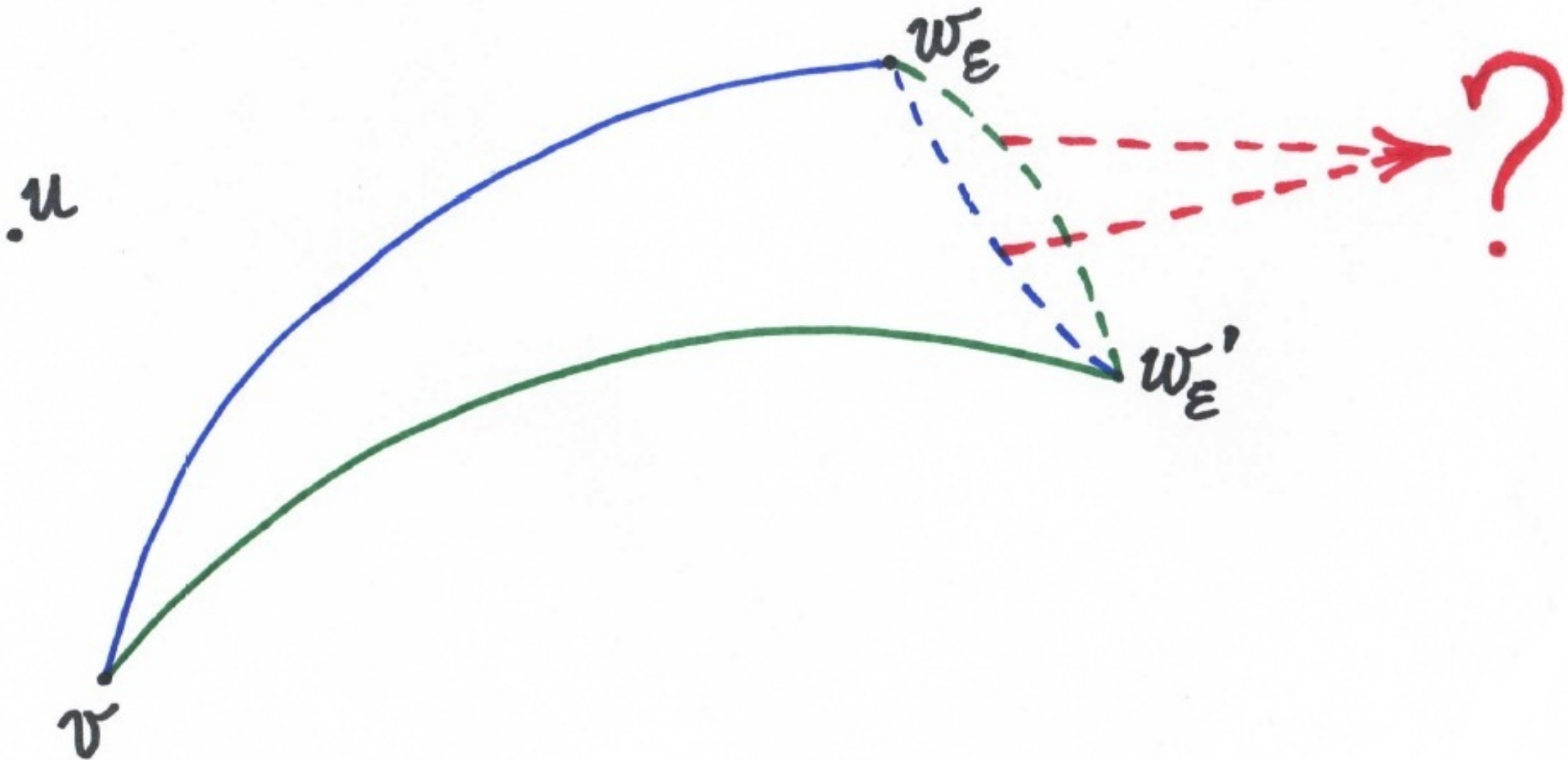
Theorem. $\forall u, v, u' \in \mathcal{U}$, $w'_\varepsilon = \gamma(1)$, $w_\varepsilon = \hat{\gamma}(1)$, where $\gamma, \hat{\gamma} : [0, 1] \rightarrow \mathbb{M}$, $\gamma, \hat{\gamma} \subset \text{Box}(u, \varepsilon)$, $u' \in \text{Box}(u, \varepsilon)$ such that $\gamma(0) = \hat{\gamma}(0) = v$ and

$$\dot{\gamma}(t) = \sum_{i=1}^N b_i(t) \widehat{X}_i^{u'}(\gamma(t)), \quad \dot{\hat{\gamma}}(t) = \sum_{i=1}^N b_i(t) \widehat{X}_i^u(\gamma(t)), \quad \int_0^1 |b_i(t)| dt < S_\varepsilon^{\deg X_i},$$

$S < \infty$, $i = 1, \dots, N$, we have

$$\max\{d_\infty^{u'}(w_\varepsilon, w'_\varepsilon), d_\infty^u(w_\varepsilon, w'_\varepsilon)\} = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0, \end{cases}$$

with $O(1)$ and $o(1)$ to be uniform as $\varepsilon \rightarrow 0$.



Application of the Estimate for Groups

• Let $b_i(t) = w_i$, $i = 1, \dots, N$

$$\Rightarrow w_\varepsilon = \exp\left(\sum_{i=1}^N w_i \widehat{X}_i^u\right)(v), \quad w'_\varepsilon = \exp\left(\sum_{i=1}^N w_i \widehat{X}_i^{u'}\right)(v)$$

(Δ') $d_\infty^p(w, v) \leq d_\infty^p(w, u) + C_{\mathcal{U}} d_\infty^p(u, v)$ is true on $\mathcal{G}^p\mathbb{M}$, $p \in \mathcal{U}$

$$\Rightarrow |d_\infty^u(v, w_\varepsilon) - d_\infty^{u'}(v, w_\varepsilon)| = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0, \end{cases}$$

$$\mathbf{If} \ u' = v \Rightarrow |d_\infty^u(v, w_\varepsilon) - d_\infty(v, w_\varepsilon)| = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0, \end{cases}$$

Thus, we can compare d_∞^u and d_∞ for points from $\mathcal{U} \subset \mathbb{M}$

Comparison of Local Geometries

- $\mathcal{U} \in \mathbb{M}$ is an arbitrary neighborhood small enough
- $d_\infty(w, v)$ and $d_\infty^u(w, v)$ are well-defined $\forall u, v, w \in \mathcal{U}$

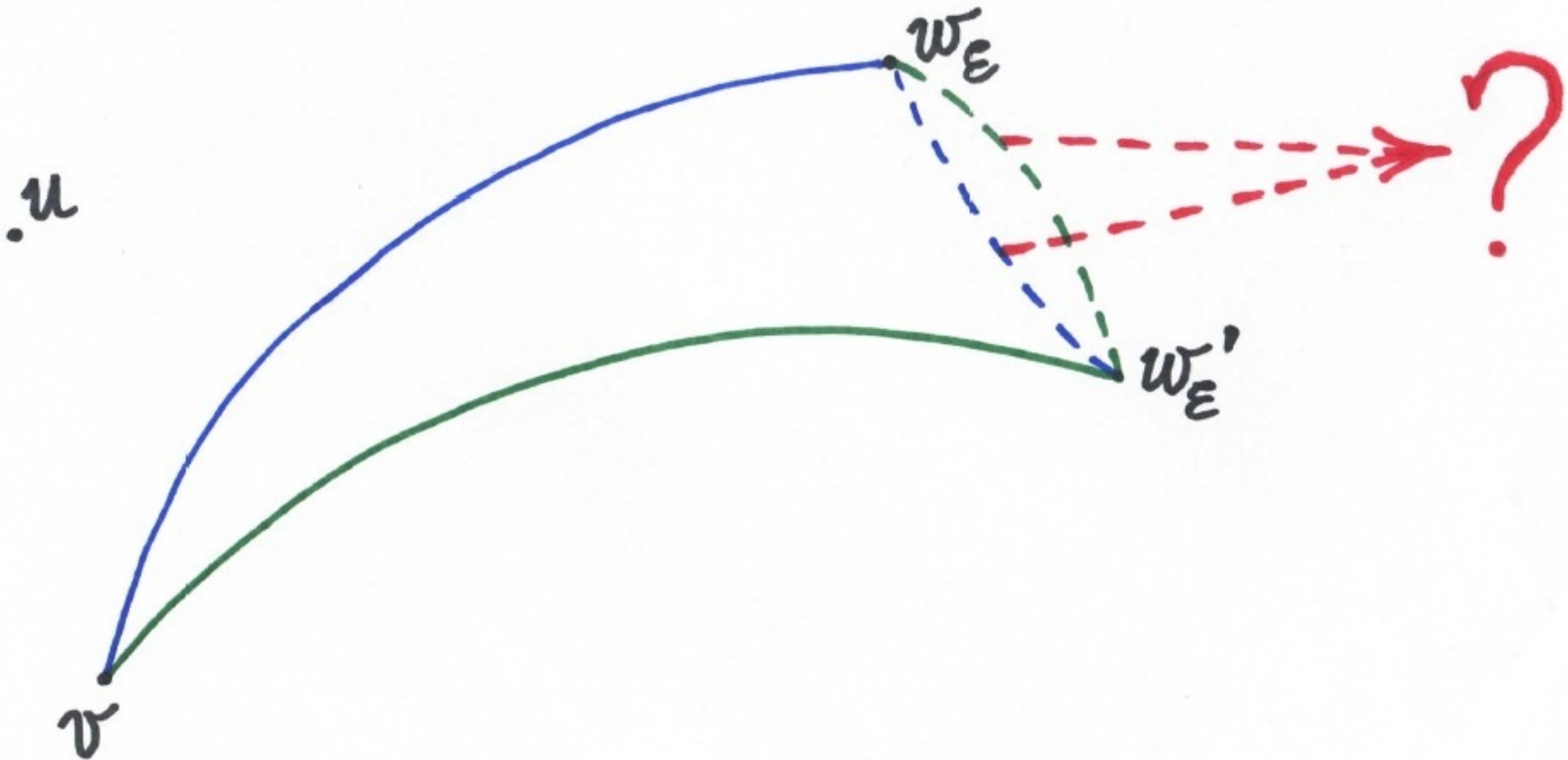
Theorem. $\forall u, v \in \mathcal{U}$, $w_\varepsilon = \gamma(1)$, $w'_\varepsilon = \hat{\gamma}(1)$, where $\gamma, \hat{\gamma} : [0, 1] \rightarrow \mathbb{M}$, $\gamma, \hat{\gamma} \subset \text{Box}(u, \varepsilon)$ such that $\gamma(0) = \hat{\gamma}(0) = v$ and

$$\dot{\gamma}(t) = \sum_{i=1}^N b_i(t) X_i(\gamma(t)), \quad \dot{\hat{\gamma}}(t) = \sum_{i=1}^N b_i(t) \widehat{X}_i^u(\gamma(t)), \quad \int_0^1 |b_i(t)| dt < S \varepsilon^{\deg X_i},$$

$S < \infty$, $i = 1, \dots, N$, we have

$$\max\{d_\infty(w_\varepsilon, w'_\varepsilon), d_\infty^u(w_\varepsilon, w'_\varepsilon)\} = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0, \end{cases}$$

with $O(1)$ and $o(1)$ to be uniform as $\varepsilon \rightarrow 0$.



Estimates for Chains (I)

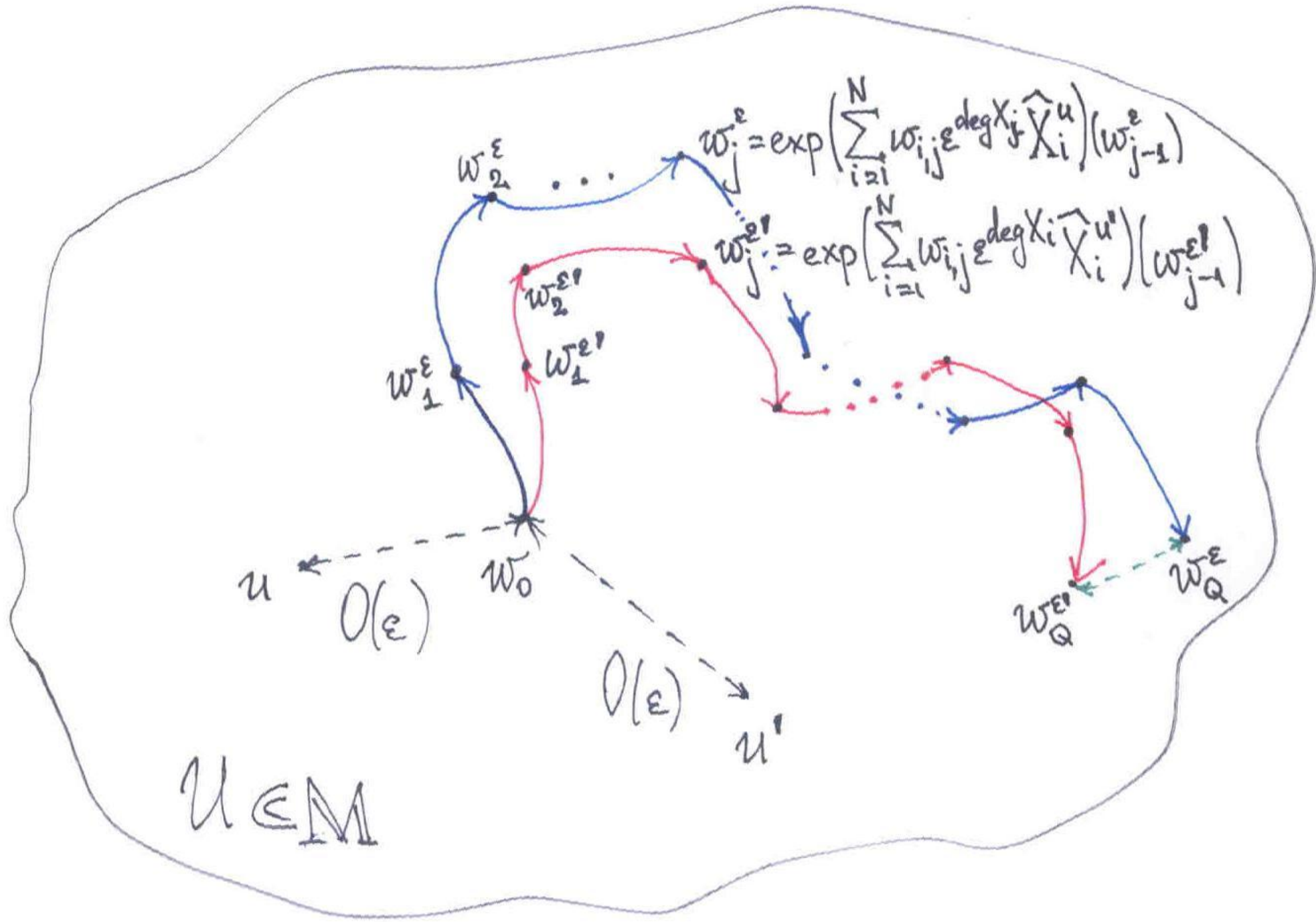
Theorem. For $Q \in \mathbb{N}$, consider $u \in \mathcal{U}$, $u', w_0 \in \text{Box}(u, \varepsilon)$ and

$$w_j^\varepsilon = \exp\left(\sum_{i=1}^N w_{i,j} \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(w_{j-1}^\varepsilon), \quad w_j^{\varepsilon'} = \exp\left(\sum_{i=1}^N w_{i,j} \varepsilon^{\deg X_i} \widehat{X}_i^{u'}\right)(w_{j-1}^{\varepsilon'})$$

$w_0^{\varepsilon'} = w_0^\varepsilon = w_0$, $j = 1, \dots, Q$. (Here $Q \in \mathbb{N}$ is such that all these points and lines lie in $\mathcal{U} \subset \mathbb{M}$ for sufficiently small $\varepsilon > 0$.) Then

$$\max\{d_\infty^u(w_Q^\varepsilon, w_Q^{\varepsilon'}), d_\infty^{u'}(w_Q^\varepsilon, w_Q^{\varepsilon'})\} = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0. \end{cases}$$

Here $O(1)$ is uniform in $u \in \mathcal{U}$, $u', w_0 \in \text{Box}(u, \varepsilon)$ and $\{w_{i,j}\}$, $i = 1, \dots, N$, $j = 1, \dots, Q$, from some compact neighborhood of 0, and depends on Q and $\{F_{\mu,\beta}^j | \mathcal{U}\}_{j,\mu,\beta}$.



Estimates for Chains (II)

Theorem. For $Q \in \mathbb{N}$, consider $u \in \mathcal{U}$, $u', w_0 \in \text{Box}(u, \varepsilon)$ and

$$w_j^\varepsilon = \exp\left(\sum_{i=1}^N w_{i,j} \varepsilon^{\deg X_i} \widehat{X}_i^u\right)(w_{j-1}^\varepsilon), \quad w_j^{\varepsilon'} = \exp\left(\sum_{i=1}^N w_{i,j} \varepsilon^{\deg X_i} X_i\right)(w_{j-1}^{\varepsilon'}),$$

$w_0^{\varepsilon'} = w_0^\varepsilon = w_0$, $j = 1, \dots, Q$. (Here $Q \in \mathbb{N}$ is such that all these points and lines lie in $\mathcal{U} \subset \mathbb{M}$ for sufficiently small $\varepsilon > 0$.) Then

$$\max\{d_\infty^u(w_Q^\varepsilon, w_Q^{\varepsilon'}), d_\infty(w_Q^\varepsilon, w_Q^{\varepsilon'})\} = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0. \end{cases}$$

Here $O(1)$ is uniform in $u \in \mathcal{U}$, $w_0 \in \text{Box}(u, \varepsilon)$ and $\{w_{i,j}\}$, $i = 1, \dots, N$, $j = 1, \dots, Q$, from some compact neighborhood of 0, and depends on Q and $\{F_{\mu,\beta}^j|_{\mathcal{U}}\}_{j,\mu,\beta}$.

Application: Rashevsky–Chow Theorem

Suppose that \mathbb{M} is a Carnot manifold

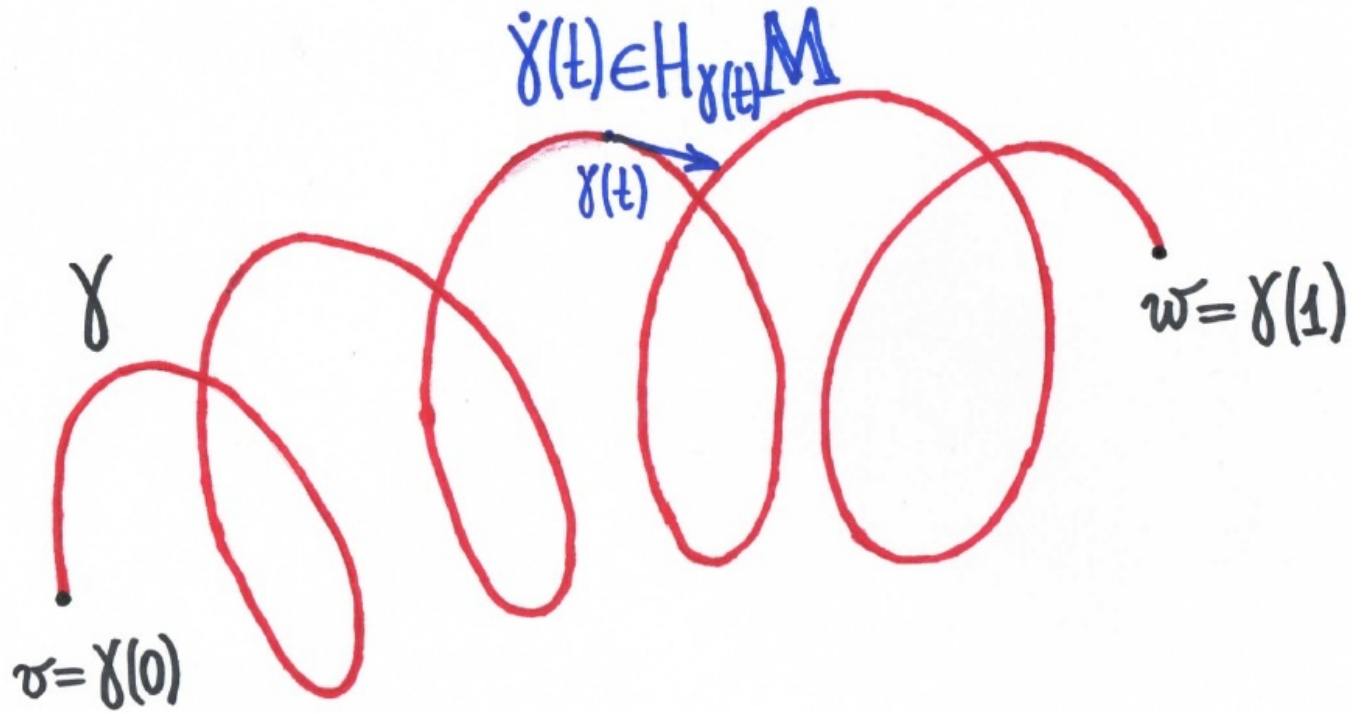
Theorem. *Any two points $x, y \in \mathbb{M}$ can be connected by a horizontal curve γ (i. e., $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0, 1]$).*

- Vodopyanov, Basalaev: C^1 -smooth case

Definition. Carnot–Carathéodory distance d_{cc} between points x and y equals

$$\inf\{\ell(\gamma) : \gamma(0) = x, \gamma(1) = y, \dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}\}$$

M



Corollaries

- d_∞ is a quasimetric
- d_∞ and d_{cc} are locally bi-Lipschitz equivalent (Ball-Box Theorem):

$$c(\mathcal{U})d_\infty(x, y) \leq d_{cc}(x, y) \leq C(\mathcal{U})d_\infty(x, y),$$

- $\dim_{\mathcal{H}} \mathbb{M} = \dim H_1 + \sum_{i=2}^M i(\dim H_i - \dim H_{i-1}) := \nu$
- Local Approximation Theorems: $u', v, w \in B_{cc}(u, \varepsilon) \Rightarrow$

$$|d_\infty(v, w) - d_\infty^u(v, w)| = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0; \end{cases}$$

$$|d_{cc}(v, w) - d_{cc}^u(v, w)| = \begin{cases} O(\varepsilon^{1+\frac{\alpha}{M}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0. \end{cases}$$

- sub-Riemannian analog of manifold's approximation by a tangent space

Weighted Carnot–Carathéodory Spaces

- Weights instead of degrees: $l_1 < \dots < l_Q$, $l_k \in \mathbb{N}$, $k = 1, \dots, Q$
- «usual» Carnot–Carathéodory space: $l_k = k$, $Q = M$
- $\{X_i\}_{i=1}^N \in C^M$: S. Selivanova, definition and properties (+non-equiregularity)
- $[H_i, H_j] \subseteq H_m$, $m = \max\{p : l_i + l_j \geq l_p\}$, $i, j = 1, \dots, Q \Leftrightarrow$
 $[X_i, X_j](v) = \sum_{k: \text{wgt } X_k \leq \text{wgt } X_i + \text{wgt } X_j} c_{ijk}(v) X_k(v) \quad (\{X_i\}_{i=1}^N \in C^1)$
- $\{c_{ijk}(u)\}_{\text{wgt } X_k = \text{wgt } X_i + \text{wgt } X_j} = \{\bar{c}_{ijk}\} \Rightarrow$ local Lie group
- comparison estimates:
$$\begin{cases} O(\varepsilon^{1 + \frac{\min\{\alpha l_1, 1\}}{l_Q}}), & \alpha > 0, \\ o(\varepsilon), & \alpha = 0, \end{cases}$$

Main Applications

- Sub-Riemannian Differentiability Theory: Rademacher- and Stepanov-type theorems on sub-Riemannian differentiability of mappings of Carnot manifolds (S. Vodopyanov)
- Geometric Measure Theory on metric structures: area formula for intrinsically Lipschitz mappings, coarea formula for C^{M+1} -smooth mappings of Carnot manifolds, criteria for minimal surfaces (M. Karmanova; S. Vodopyanov and M. Karmanova)
- Geometry of non-equiregular (and weighted) Carnot–Carathéodory spaces (S. Selivanova)
- Rashevsky–Chow Theorem for C^1 -smooth case \Rightarrow problem on attainable sets (S. Basalaev, S. Vodopyanov)

Sub-Riemannian Differentiability

Definition [S. Vodopyanov]. Let \mathbb{M} and $\widetilde{\mathbb{M}}$ be CC-spaces, and let $D \subset \mathbb{M}$. A mapping $\varphi : (D, d_\infty) \rightarrow (\widetilde{\mathbb{M}}, \widetilde{d}_\infty)$ is *hc-differentiable* at $u \in \mathbb{M}$, if there exists a horizontal homomorphism

$$L_u : (\mathcal{G}^u, d_\infty^u) \rightarrow (\mathcal{G}^{\varphi(u)}, d_\infty^{\varphi(u)})$$

of local Carnot groups such that

$$\widetilde{d}_\infty(\varphi(w), L_u(w)) = o(d_\infty(u, w)), \quad D \cap \mathcal{G}^u \ni w \rightarrow u.$$

- $\varphi : D \rightarrow \widetilde{\mathbb{G}}, D \subset \mathbb{G} \Rightarrow \mathcal{P}$ -differentiability [Pansu]
- *hc*-differential of φ at u is denoted by $\widehat{D}\varphi(u)$

Theorem (Vodopyanov). Let \mathbb{M} be a Carnot manifold, $D \subset \mathbb{M}$ be a measurable set, and $\widetilde{\mathbb{M}}$ be a Carnot–Carathéodory space. Every intrinsically Lipschitz mapping $\varphi : D \rightarrow \widetilde{\mathbb{M}}$ is *hc*-differentiable almost everywhere.

Sub-Riemannian Area Formula

Classical result: $\varphi : D \rightarrow \mathbb{R}^m$, $D \subset \mathbb{R}^n$, $n \leq m$

$$\int_D f(y) \sqrt{\det(D\varphi(y)^* D\varphi(y))} d\mathcal{H}^n(y) = \int_{\mathbb{R}^m} \sum_{y: y \in \varphi^{-1}(x)} f(y) d\mathcal{H}^n(x)$$

Sub-Riemannian Result: \mathbb{M} is a Carnot manifold, $\widetilde{\mathbb{M}}$ is a CC-space, $D \subset \mathbb{M}$, $\varphi \in \text{Lip}^{SR}(D, \widetilde{\mathbb{M}})$, $\dim H_1 \leq \dim \widetilde{H}_1$;

$$\int_{\mathbb{M}} f(y) \sqrt{\det(\widehat{D}\varphi(x)^* \widehat{D}\varphi(x))} d\mathcal{H}^\nu(y) = \int_{\widetilde{\mathbb{M}}} \sum_{y: y \in \varphi^{-1}(x)} f(y) d\mathcal{H}^\nu(x)$$

Sub-Riemannian Coarea Formula

Classical result: $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^k, n \geq k$

$$\int_{\mathbb{R}^n} f(x) \sqrt{\det(D\varphi(x)D\varphi(x)^*)} d\mathcal{H}^n(x) = \int_{\mathbb{R}^k} d\mathcal{H}^k(z) \int_{\varphi^{-1}(z)} f(u) d\mathcal{H}^{n-k}(u)$$

Sub-Riemannian Result: \mathbb{M} is a Carnot manifold, $\tilde{\mathbb{M}}$ is a CC-space, $\varphi \in C^{M+1}(\mathbb{M}, \tilde{\mathbb{M}})$ is a contact mapping, $\dim H_1 \geq \dim \tilde{H}_1$, $\dim H_i - \dim H_{i-1} \geq \dim \tilde{H}_i - \dim \tilde{H}_{i-1}, i = 2, \dots, M$;

$$\int_{\mathbb{M}} \mathcal{J}_{\tilde{N}}^{SR}(\varphi, x) f(x) d\mathcal{H}^\nu(x) = \int_{\tilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(z) \int_{\varphi^{-1}(z)} f(u) d\mathcal{H}^{\nu-\tilde{\nu}}(u),$$

where $\mathcal{J}_{\tilde{N}}^{SR}(\varphi, x) = \sqrt{\det(\widehat{D}\varphi(x)\widehat{D}\varphi(x)^*)} \cdot C_\omega$

C_ω depends on factors in the definitions in Hausdorff measure

Main Publications

1. *Karmanova M. B.* Fine properties of basis vector fields on Carnot–Carathéodory spaces under minimal assumptions on smoothness // *Sib. Math. Zh.*, 2014. V. 55, № 1. P. 87–99.
2. *Karmanova M. B.* Area formula for Lipschitz mappings of Carnot–Carathéodory spaces // *Izvestiya RAS*, 2014 (accepted)
3. *Karmanova M., Vodopyanov S.* On Local Approximation Theorem on Equiregular Carnot-Carathéodory Spaces // *Proceedings of INDAM "Meeting on Geometric Control and Sub-Riemannian Geometry"*, Cortona, May 2012. Springer INdAM Series, 2014 (accepted)
4. *Karmanova M., Vodopyanov S.* A Coarea Formula for Smooth Contact Mappings of Carnot-Carathéodory Spaces // *Acta Applicandae Mathematicae*, 2013. V. 128, № 1. P. 67–111.
5. *Karmanova M.B.* Graphs of Lipschitz functions and minimal surfaces on Carnot groups // *Sib. Math. Zh.*, 2012. V. 53, № 4. P. 839–861.
6. *Karmanova M., Vodopyanov S.* An Area Formula for Contact C^1 -Mappings of Carnot Manifolds // *Complex Variables and Elliptic Equations*, 2010. V. 55, Issue I-III. P. 317–329.

THANK YOU!

