Shadow prices
in infinite horizon control problems

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Original infinite horizon problem

$$
\mathcal{P}\begin{cases} \dot{x} = f(t,x,u), & t \in [0,\infty), \\ x(0) = \xi_*, & x \in \mathbb{R}^m, \\ u(t) \in U(t), & \\ \text{minimize} & \int_0^\infty g\big(t, x(u;t), u\big)dt \end{cases}
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- The setmap U is compact-valued map; it has a Borel measurable graph; it is integrally bounded on each compact subset;
- The locally Lipshitz continuous on x Carathéodori functions f, g, $\frac{\partial f}{\partial x}, \frac{\partial g}{\partial x}$ are integrally bounded on any compact;
- f has the sublinear growth with respect to x.

The value function and payoff function

For each $\theta \geq 0, y \in \mathbb{R}^m, T > \theta$ put

$$J(\theta, \xi, u; T) \triangleq \int_\theta^T g\big(t, x(\theta, y, u; t), u(t)\big) dt.$$

$$\mathcal{P}(\theta, \xi; T) \begin{cases} \dot{x} = f(t, x, u), & t \in [\theta, T], \\ x(\theta) = y, & x \in \mathbb{R}^m, \\ u(t) \in U(t), & \\ \text{minimize} & J(\theta, \xi, u; T) \end{cases}$$

By $V(\theta, \xi; T)$ denote the infimum of $J(\theta, \xi, u; T)$, the optimal value of $\mathcal{P}(\theta, \xi; T)$.

Let u^* be an optimal control for $\mathcal{P}(0, \xi_*; T)$ for some $T > 0$

shortcut problem: minimize $J(0, \xi_*, u; T)$

Then for $\lambda^* = 1$ and some co-state arc p^*
the Pontryagin Maximum Principle holds:

(PMP):

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(PMP): Pontryagin rule	$\mathcal{H}[t] = H\big(t,x(t),p(t),\lambda,u^*(t)\big)\ a.e.;$
dynamics equation	$\dot{x} = f(t,x,u^*)\ a.e.,\ x(0)=\xi_*;$
adjoint equation	$\dot{p}(t) = -\dfrac{\partial H}{\partial x}(t,x,p,\lambda,u^*)\ a.e.;$
transversality condition	$p(T)=0;$
sensitivity relations	$-p(0)\in\partial_x V(0,x(0);T),$
	$\big(\mathcal{H}[t],-p(t)\big)\in\partial V(t,x(t);T)\ a.e.;$
with the Hamiltonian function	$H \triangleq pf\big(t,x,u\big)-\lambda g\big(t,x,u\big),$
the maximized Hamiltonian	$\mathcal{H}[t] \triangleq \max\limits_{v\in U(t)} H\big(t,x(t),p(t),\lambda,v\big).$

Co-state arc $p(0)$ as shadow price $-\frac{dV}{dx}(0,\xi_*;T)$;

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The difference of optimal values for a small additional production factor δ is

$$V(0,x(0);T) - V(0,x(0)+\delta;T) \approx -\frac{dV}{dx}(0,\xi_*;T)\delta = p^*(0)\delta.$$

The payment for a small additional production factor δ is $c\delta$.
Cost c of less than $p^*(0)$ will be profit.

Suppose u^* is an optimal control for $\mathcal{P}(0, \xi_*; \tau_n)$ for all n

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\text{nontriviality condition:} \quad & \lambda + ||p(0)|| > 0;
\end{aligned}$$

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transversality condition??	$\lim_{T\to\infty} p(T) = 0$
	$\lim_{T\to\infty} H[T] = 0$
	Such conditions may fail
	or may be trivial
Sensitivity relations?	$V(t,x;\infty)$ may not exist

Typical example [Seierstad 1999, Ex. 10.2]

$$\dot{x} = ux,\ \ x(0) = 1, u \in [1/2, 1],$$
$$\text{minimize} \int_0^\infty -xe^{-2t}dt$$

Here $u^* \equiv 1$ is optimal control, $x^* = e^t$ is the optimal motion.

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adjoint equation: $\lambda^* = 1,\ \dot{p} = xp + xe^{-2t}$

any solution: $p_C(t) = Ce^{-2t} + (1 - C)e^{-t}, \forall C \in \mathbb{R}$

transversality condition: $\lim_{t\to\infty} p_C(t) = 0 \qquad \forall C \in \mathbb{R}$.

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This condition is useless.

Hence the need to find a new necessary boundary condition.

Seierstad approach

shortcut problem:
$$\dot{x} = ux,\; x(0) = 1, u \in [1/2, 1]$$
$$\int_0^{\ln n} -xe^{-2t} dt \to \min .$$

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shortcut PMP: $$\dot{p} = xp + xe^{-2t},\ \ p_n(\ln n) = 0, \lambda_n = 1,$$

solution of shortcut PMP: $$p_n(t) = \frac{ne^{-2t} - e^{-t}}{n-1},\ \ \lambda_n = 1,\ \ x = e^t$$

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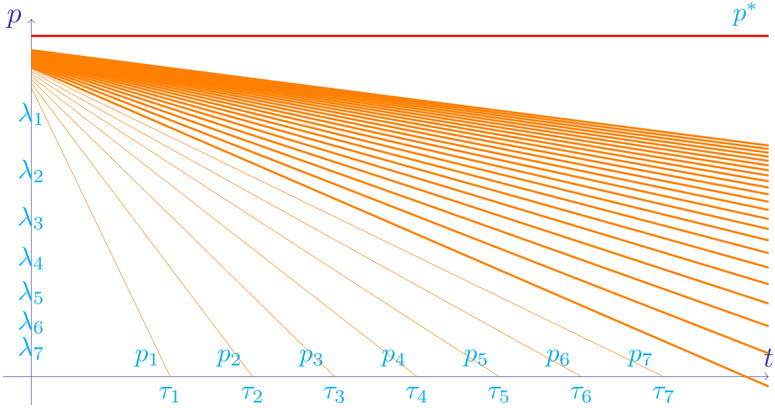
Put
$$p^*(t) = \lim_{n\to\infty} p_n(t) = e^{-2t},\ \ \lambda^* = 1.$$

Such pointwise limit is a solution of PMP under additional assumptions (see [Seierstad 1999; Th. 6.1]).

the limit (p^*, λ^*) of solution (p_n, λ_n) is solution of PMP



Necessary conditions for strong optimal control

The easy theorem. (u^* is τ-strong optimal control)
Suppose u^* is an optimal control for $\mathcal{P}(0, \xi_*; \tau_n)$ for all $n \in \mathbb{N}$.
Then there exists a τ-vanishing solution (p^*, λ^*) of PMP.

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Then there exists a τ-vanishing solution (p^*, λ^*) of PMP.
In particular, (p^*, x^*, λ^*) satisfies

$$\begin{aligned} \text{(PMP): Pontryagin rule} \qquad & \mathcal{H}[t] = H\big(t, x^*(t), p^*(t), \lambda^*, u^*(t)\big) \ a.e.\ t > 0 \\ \text{dynamics equation} \qquad & \dot{x} = f(t, x^*, u^*), x(0) = \xi_*; \\ \text{adjoint equation} \qquad & \dot{p}^*(t) = -\frac{\partial H}{\partial x}(t, x^*, p^*, \lambda^*, u^*). \end{aligned}$$

See more on this and other necessary conditions of τ-strong optimality in [Khlopin Izhevsk 2013]

τ-vanishing solution of PMP

Call a nontrivial solution (p^*, λ^*) of PMP associated with (x^*, u^*) τ-vanishing if (p^*, x^*, λ^*) is a pointwise limit of solutions (p_n, x_n, λ_n) of boundary value problems

$$\begin{aligned} p_n(\tau'_n) &= 0, \qquad \lambda_n = const > 0 \qquad \text{(system of} \\ \dot{x}_n(t) &= f(t, x_n(t), u^*(t)), \qquad \text{shortcut PMP)} \\ \dot{p}_n(t) &= -\frac{\partial H}{\partial x}(x_n(t), t, u^*(t), \lambda_n, p_n(t)); \end{aligned}$$

here, τ' is a test subsequence ($\tau' \subset \tau$).

PMP for weakly uniformly overtaking optimal control

The main theorem (u^* is weakly uniformly overtaking optimal).
Let u^* be a ε_n-optimal control for $\mathcal{P}(0, \xi_*; \tau_n)$
for all $n \in \mathbb{N}$, here $\varepsilon_n \downarrow 0, \tau_n \uparrow \infty$
Then there exists a τ-vanishing solution (p^*, λ^*) of PMP.

Ingredients of proof:

- Alaoglu–Warga compactification of the set of admissible controls;
- penalty functions that guarantee some stability for optimal trajectories of auxiliary problems;
- semicontinuity of PMP.

See more in [Khlopin JDCS 2013]

Suppose that V is smooth, and there exist a finite limit

$$Sh.Pr \triangleq \lim_{\xi \to \xi_*, T \to \infty} -\frac{\partial V}{\partial \xi}(0, \xi; T).$$

Then the problem is normal (no vanishing solution of PMP with $\lambda = 0$),

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Let there exist a finite limit

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Then the problem is normal (no vanishing solution of PMP with $\lambda = 0$), there exists a unique vanishing solution (λ^*, p^*) with $\lambda^* = 1$, and

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See also in [Aseev, 2009],[Sagara 2010],[Aseev,Besov,Kryazhimskii 2012]

On good case

(0)

$$\exists S \triangleq \lim_{\xi\to\xi_*, T\to\infty} -\frac{\partial J}{\partial \xi}(0,\xi,u^*;T) \in \mathbb{R}^m \ \Rightarrow$$

$\Rightarrow$ (1) The problem $\mathcal{P}$ is normal,

$\Rightarrow$ (2) there exist a unique vanishing solution of form $(1, p^*)$,

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$$\exists S \triangleq \lim_{T\to\infty} -\frac{\partial J}{\partial \xi}(0, \xi_*, u^*; T) \in \mathbb{R}^m;$$

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(Definition of vanishing solution)

A nontrivial solution (p^*, λ^*) of PMP associated with (x^*, u^*) is called τ-vanishing if for some test subsequence τ', (p^*, x^*, λ^*) is a pointwise limit of solutions (p_n, x_n, λ_n) of boundary value problems

$$\begin{aligned} p_n(\tau'_n) &= 0, \qquad \lambda_n = const > 0 \\ \dot{x}_n(t) &= f(t, x_n(t), u^*(t)), \\ \dot{p}_n(t) &= -\frac{\partial H}{\partial x}(x_n(t), t, u^*(t), \lambda_n, p_n(t)); \end{aligned}$$

In general, $x_n \neq x^*$.

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In general, $x_n \not\equiv x^*$. And so, even if

- u^* is uniformly overtaking optimal;
- the problem is normal;
- a vanishing solution of the PMP is unique, accurate to a positive factor, and independent of the choice of τ;
- the function $\frac{dV}{dx}(0, \xi_*; T)$ has a finite limit as $T \to \infty$.

The situation $x_n \equiv x^*$

Define a normalized fundamental matrix A as the solution of Cauchy problem

$$\frac{dA(t)}{dt} = \frac{\partial f}{\partial x}(t, x^*(t), u^*(t))A(t), \;\; A(0) = 1,$$

Aseev–Kryazhimskii formula $p^*(0) = \int_0^\infty \frac{\partial g}{\partial x}(t, x^*(t), u^*(t))\, A(t)dt$

points to a vanishing solution

if $x_n \equiv x^*$, $\lambda^* = 1$, and this integral converges.In particular,
if assumptions for any statements of below are satisfied

- [Seierstad 1999,Theorem 6.1],
- [Aseev and Veliov 2012],
- [Khlopin ISDG 2013,Sect. 5],
- [Aseev, Besov and Kryazhimskii 2012; Sect. 4],
- [Aseev and Kryazhimskii 2007; Sect. 4],
- [Aubin and Clarke1979].

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Suppose that $\lambda^* = 0$; then there exists no finite limit

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But, for some subsequence $\tau' \subset \tau$

$$p_i^*(0) \Big/ p_k^*(0) = \lim_{\xi \to \xi_*, n \to \infty} \frac{\partial V}{\partial \xi_i}(\xi, 0; \tau'_n) \Big/ \frac{\partial V}{\partial \xi_k}(\xi, 0; \tau'_n), \qquad \forall i$$

here

$$p^*(0) = (p_1^*(0), p_2^*(0), \ldots, p_m^*(0)), \ |p_k^*(0)| = \max_i |p_i^*(0)|.$$

In general,

$$\exists \lim_{T \to \infty} -\frac{\partial V}{\partial \xi}(0, \xi_*; T) \in \mathbb{R}^m$$

does not implies $\lambda^* > 0$.

The payoff function: general situation

$$\partial^1_x \hat{J}(0,\xi,u^*;\infty_\tau) \triangleq \Big\{ \zeta \in \mathbb{R}^m \;\Big|\; \exists \xi_n \in \mathbb{R}^m, t_n = \tau_k, \zeta_n = \frac{\partial \hat{J}}{\partial \xi}(0,\xi_n,u^*;t_n),$$

(generalized gradient)

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The payoff function: general situation

$$\partial_x^1 \hat{J}(0,\xi,u^*;\infty_\tau) \triangleq \Big\{\zeta\in\mathbb{R}^m \ \Big|\ \exists \xi_n\in\mathbb{R}^m, t_n=\tau_k, \zeta_n=\frac{\partial \hat{J}}{\partial \xi}(0,\xi_n,u^*;t_n),$$

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Let u^* be weakly uniformly optimal; then there exist vanishing solution (λ^*,p^*) of PMP and test sequence τ such that for $\hat{J}\equiv -J$

$$\text{or } \lambda^*=1,\qquad p^*(0)\in\partial_x^1\hat{J}(\xi_*,u^*;\infty_\tau),\qquad \partial_x^0\hat{J}(\xi_*,u^*;\infty_\tau)=\varnothing;$$
$$\text{either } \lambda^*=0,\qquad p^*(0)\in\partial_x^0\hat{J}(\xi_*,u^*;\infty_\tau),\qquad \partial_x^1\hat{J}(\xi_*,u^*;\infty_\tau)=\varnothing.$$

The value function: general situation

$$\partial_x^1 \hat{V}(\theta, \xi; \infty_\tau) \triangleq \Big\{ \zeta \in \mathbb{R}^m \;\Big|\; \exists \xi_n \in \mathbb{R}^m, t_n = \tau_k, \zeta_n = \frac{\partial \hat{V}}{\partial \xi}(\theta, \xi_n; t_n), \xi_n \to \xi_*, t_n \to \infty, \zeta_n \to \zeta \Big\};$$

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Moreover, if V is smooth; then for $\hat{V} \equiv -V$,

or $\lambda^* = 1, \quad p^*(0) \in co\, \partial_x^1 \hat{V}(0, \xi_*; \infty_\tau), \quad \partial_x^0 \hat{V}(0, \xi_*; \infty) = \varnothing;$

either $\lambda^* = 0, \quad p^*(0) \in co\, \partial_x^0 \hat{V}(0, \xi_*; \infty_\tau), \quad \partial_x^1 \hat{V}(0, \xi_*; \infty) = \varnothing.$

Mayer problem with state constraints

$$\mathcal{M}[0,\xi_*,T]\begin{cases} \dot{x} = f(t,x,u), & a.e., x(0)=\xi_*, \\ h(t,x(t)) \le 0 & \\ u(t) \in U(t), & a.e. \\ \text{minimize} & G(x(T)) \text{ for large } T \end{cases}$$

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The value function V is

$$V(\theta,\xi;T)\triangleq\begin{cases} \inf \quad \text{for } \mathcal{M}[\theta,\xi;T] & \text{if there exists admissible control;} \\ +\infty & \text{otherwise} \end{cases}$$

$$\hat{V}(\xi;T)\triangleq\begin{cases} -V(0,\xi;T) \text{ if } h(0,\xi)\leq 0; \\ +\infty \text{ if } h(0,\xi)>0. \end{cases}$$

The assumptions

- The setmap U is compact-valued map; it has a Borel measurable graph; it is uniformly bounded on each compact subset;
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- (uniform inward pointing condition) (see [Soner,1986])

$$\exists \alpha, \beta > 0 \ \ |h(t,x)| < \alpha \ \Rightarrow \ \frac{\partial h}{\partial t}(t,x) + \min_{u \in U(t)} \frac{\partial h}{\partial x}(t,x) f(t,x,u) \leq -\beta.$$

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Suppose that $\{\partial V(t,y;T) \,|\, h(t,y) < 0, t \leq T\}$ are uniformly bounded.

Let u^* be a optimal control for problem $\mathcal{M}(0, \xi_*; T)$
Then for some true co-state arc p^*, pseudo co-state arc q^*,
and measure $\mu^* \in NBV^+[0;T]$, $supp\ \mu^* \subset \{t \,|\, h(x^*(t)) = 0\}$,
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(PMP): Pontryagin rule $\mathcal{H}[t] = H\big(t, x(t), q(t), u^*(t)\big), a.e.$

dynamics equation $\dot{x} = f(t,x,u^*)\, a.e., x(0) = \xi_*;$

adjoint equation $-\dot{q}(t) = \dfrac{\partial H}{\partial x}(t,x,p,u^*)\, a.e.;$

$$p(t) \triangleq q(t) + \int_{[0,t]} \nabla h_x(s,x(s))d\mu(s), \forall t \in [0,$$

sensitivity relations $p(0) \in \partial_x \hat{V}(\xi_*;T)$

$$\big(\mathcal{H}[t], -p(t)\big) \in co\,\partial^{int} V(t,x(t);T)\, a.e.;$$

transversality condition $-p(T) \in \partial_x G(T,x(T));$

with Hamiltonian $H(t,x,p,u) \triangleq p\, f(t,x,u),$

$$\mathcal{H}[t] \triangleq \sup_{v \in U(t)} H(t,x(t),p(t),v).$$

(Present theorem) Let u^* be a ε_n-optimal control

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$$\big(\mathcal{H}[t], -p(t)\big) \in \overline{co}\, \partial^{int} V(t, x(t); \infty_\tau)\, a.e.;$$

here $\partial^{int} V(t, \xi; \infty_\tau) \triangleq \{\zeta \,|\, \exists (\zeta_n, t_n, \xi_n, T_n) \to (\zeta, t, \xi, \infty)$

such that $\zeta_n \in \partial V(t_n, \xi_n, T_n), h(t_n, \xi_n) < 0, T_n = \tau_k\}.$

Moreover, there exists a sequence (p_n, q_n, x_n, μ_n) and subsequence τ' of subsequence τ such that

- $\mu^*|_K$ is a $*$-weak limit of $\mu_n|_K \in NBV^+[K]$ on any compact interval K,
- (p^*, q^*, x^*) is a uniform limit of (p_n, q_n, x_n) on any compact interval;
- each (p_n, q_n, x_n) satisfies its boundary value problem

dynamics equation $\quad \dot{x}_n = f(t, x_n(t), u^*(t));$ (system of PMP):

adjoint equation $\quad -\dot{q}_n(t) = \frac{\partial H}{\partial x}(t, x_n(t), p_n(t), u^*(t)) \ a.e.,$

$$p_n(t) \triangleq q_n(t) + \int_{[0,t]} \nabla h_x(s, x_n(s)) d\mu_n(s) \, \forall t;$$

with boundary condition

$$-p_n(\tau'_n) \in \partial_x G(\tau'_n, x_n(\tau'_n))$$

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The case $h(0, \xi_*) = 0, \nabla h_x(0, \xi_*) \neq 0.$

For $\mu \stackrel{\triangle}{=} \delta_{t=0};\ p(0) = -\nabla h_x(0, \xi_*), p(t) = 0$ if $t > 0.$

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any vanishing solution satisfies $p(t) \neq 0$ a.a. if for some $\alpha, \beta > 0$

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In [Oliveira,Silva 2009] see on nontrivial phase constraints PMP with Michel transversality condition without sensitivity conditions;

In [Pereira,Silva 2011] see on nontrivial phase constraints PMP with some transversality condition and sensitivity condition for shadow price.

The reference list. The part I

Arutyunov A.V., Optimality conditions: Abnormal and Degenerate Problems, Mathematics and its Application, Kluwer, 2000.

Arutyunov, A.V. Aseev, S.M. Investigation of the degeneracy phenomenon of the maximum principle for optimal control problems with state constraints, *SIAM J. Control Optim.* 35, 930–952, 1997

Arutyunov A.V.,Karamzin D.Y.,Pereira F.L., The Maximum Principle for Optimal Control Problems with State Constraints by R.V. Gamkrelidze: Revisited. *J. Optim. Theory Appl.* 149, 474–493, 2011.

Aseev, S.M., Infinite-horizon optimal control with applications in growth theory, MAKS Press, Moscow, 2009.

Aseev, S.M., Kryazhimskii, A.V. The Pontryagin Maximum Principle and transversality conditions for a class of optimal control problems with infinite time horizons, *SIAM J. Control Optim.* 43, 1094–1119, 2004

The reference list. The part II

Aseev, S.M., Kryazhimskii, A.V. The Pontryagin Maximum Principle and problems of optimal economic growth, *Proc. Steklov Inst. Math.* 257:1, 1–255, 2007

Aseev, S.M., Besov, K.O., Kryazhimskii, A.V. Infinite-horizon optimal control problems in economics, *Russ. Math. Surv.*, 67:2, 195-253, 2012.

Aseev, S.M., Veliov, V.M. Maximum Principle for infinite-horizon optimal control; problems with dominating discount, *Dynamics of Continuous, Discrete and Impulsive Systems, Series B.* 19:1–2, 43–63, 2012.

Aseev, S.M., Veliov, V.M. Needle Variations in Infinite-Horizon Optimal Control *IIASA Interim Rept.* IR-2012-04. 2012

Aubin, J.P., Clarke, F.H. Shadow Prices and Duality for a Class of Optimal Control Problems, *SIAM J. Control Optim.* 17, 567-586, 1979.

Aubin, J.-P., Ekeland I. Applied nonlinear analysis, Wiley, 1984.

Bettiol, P., Vinter, R.B. Sensitivity interpretations of the costate variable for optimal control problems with state constraints, *SIAM J. Control Optim.* 48, 3297–3317, 2010.

The reference list. The part III

Bettiol, P., Bressan, A., Vinter, R.B. On Trajectories Satisfying a State Constraint: $W^{1,1}$ Estimates and Counter-Examples, *SIAM J. Control Optim.* 48, 4664–4679, 2010.

Carlson, D.A., Haurie, A.B., Leizarowitz A., Infinite Horizon Optimal Control. Deterministic and Stochastic Systems. Springer, Berlin, 1991.

Ferreira M.M.A., Vinter, R.B., When is the maximum principle for state-constrainted problems degenerate?, *J. Math. Anal. Appl.*, 187, 432-467, 1994.

Frankowska, H., Mazzola M. On relations of the adjoint state to the value function for optimal control problems with state constraints *Nonlinear Differ. Equ. Appl.* 20, 361–383, 2013

Halkin, H. Necessary Conditions for Optimal Control Problems with Infinite Horizons, *Econometrica*, 42, 267–272, 1974.

Hammond, P.J., Kennan, J. Uniformly optimal infinite-horizon plans, *International Economic Review*, 20, 283–296, 1979.

The reference list. The part IV

Khlopin, D.V. Necessity of vanishing shadow price in infinite horizon control problems. *J.Dyn.&Con.Sys*, 19, 4, 519-552, 2013

Khlopin, D.V. On necessary boundary conditions for strongly optimal control in infinite horizon control problems. *Bull. of Udmurt. Univ.*, 1, 49–58, 2013

Lopes, S.O., Fontes F.A., de Pinho M., On the Relation Between Inward Pointing Velocity Conditions *Discrete and Continuous Dynamical Systems Series A (DCDS-A)*, 29, 559–575, 2011.

Malanowskii K. On Normality of Lagrange Multipliers for State Constrained Optimal Control Problems *Optimization*, 52, 1, 75–91, 2003.

Michel P. On the transversality condition in infinite horizon optimal problems. *Econometrica*, 50, 975-984, 1982.

Oliveira V.A., Silva, G.N. Optimality conditions for infinite horizon control problems with state constraints, *Nonlinear Analysis* 71, 1788-1795, 2009.

Pereira F.L., Silva G.N. Necessary Conditions of Optimality for State Constrained Infinite Horizon Differential Inclusions In *50th IEEE Conference on Decision and Control and European Control Conference*, 2011

The reference list. The part V

Pontryagin, L.S., Boltyanskij, V.G., Gamkrelidze, R.V., Mishchenko, E.F. Mathematical Theory of Optimal Processes, N.Y., J. Wiley, 1962.

Sagara, N. Value functions and transversality conditions for infinite-horizon optimal control problems, *Set-Valued Var. Anal.*, 18, 1, 1-28, 2010.

Seierstad, A. Necessary conditions for nonsmooth, infinite-horizon optimal control problems, *J. Optim. Theory Appl.*, 103:1, 201-230, 1999.

Soner, H.M. Optimal control with state-space constraint I, *SIAM J. Control Optim.* 24, 552-561, 1986

Vinter, R.B. Optimal Control, Birkhaüser, Boston, 2000.