Subriemannian And Riemannian Affinor Structures On The Lie Algebroids

Eugene Kornev

Kemerovo State University

Let M be a C^∞-manyfold with dimension n and E be a vector bundle over M with rank r. A vector bundle E is called the Lie Algebroid if in the sections space $C^\infty(E)$ is given the bracket operation $[\sigma, \tau]$ for any sections $\sigma, \tau \in C^\infty(E)$, and exists a smooth mapping called Anchor $A : C^\infty(E) \to TM$ that satisfies the following conditions:

- for any sections $\sigma, \tau \in C^\infty(E), [A\sigma, A\tau] = A[\sigma, \tau]$;
- for any function $f \in C^\infty(M)$ and any sections $\sigma, \tau \in C^\infty(E)$,

$$[\sigma, f\tau] = (A\sigma)(f)\tau + f[\sigma, \tau].$$

The trivial examples of Lie Algebroids are TM with vector fields bracket, where A is the Identity Mapping, and T^*M with zero bracket, where A is the Duality Mapping.

The Lie Algebroid structure on vector bundle E allows us define the action of any section σ to function f as $\sigma(f) = (A\sigma)(f)$.
Now, we can define the Exterior Differential of 1-form α over E by standart way:

$$d\alpha(\sigma, \tau) = \sigma(\alpha(\tau)) - \tau(\alpha(\sigma)) - \alpha([\sigma, \tau]),$$
$$\forall \sigma, \tau \in C^\infty(E).$$

A radical of 1-form α is subbundle $\mathrm{rad}\,\alpha$ so that for any section $\sigma \in C^\infty(\mathrm{rad}\,\alpha), I_\sigma d\alpha = 0$, where I_σ is the operator of interior product with section σ. 1-form α is called regular if a vector subbundle $\mathrm{rad}\,\alpha$ has a constant rank.
If 1-form α is regular and M is a paracompact manyfold then the Lie Algebroid E reduced into Whitney sum $D \oplus \mathrm{rad}\,\alpha$, where D is a vector subbundle orthogonal to $\mathrm{rad}\,\alpha$ with respect to some Riemannian metric in E. This vector subbundle D is called Work Subbundle.

For radical of regular 1-form α can be proved the following result:

Theorem 1.

Let α be a regular unclosed 1-form over the Lie Algebroid E with rank $r \geq 3$. Then

$$0 \leq \operatorname{rank}(\operatorname{rad}\alpha) \leq r - 2.$$

More over:

(1) If E has even rank then $\operatorname{rad}\alpha$ has even rank also.

(2) If E has odd rank then $\operatorname{rad}\alpha$ has odd rank also.

This theorem follows the following important fact:

Corollary 2.

If α is a regular unclosed 1-form over the Lie Algebroid with arbitrary rank not less than 3 then a Work Subbundle $D = \operatorname{rad}\alpha^{\perp}$ always has a even rank.

Definition 3.

Affinor Structure tamed by a regular 1-form α over the Lie Algebroid E is the continuous field of fibres endomorphisms Φ that satisfies the following conditions:

(1) $\ker \Phi = \operatorname{rad} \alpha$.

(2) $\Phi D = D, \Phi^2|_D = -\operatorname{id}$.

(3) $\Phi^* d\alpha = d\alpha$.

(4) For each $\sigma \in D$ $d\alpha(\sigma, \Phi\sigma) \geq 0$.

Where $D = \operatorname{rad} \alpha^\perp$ is a work subbundle.

This definitions along with corrollary 2 follows that Affinor Structure Φ restriction to work subbundle D is the complex structure in D tamed by 2-form $d\alpha|_D$.

An affinor definition follows that 2-form

$$d\alpha_{\Phi}(\sigma,\tau) = d\alpha(\sigma,\Phi\tau) \quad \forall \sigma,\tau \in D$$

is a Riemannian Metric in the work subbundle D. Hence, pair $D, d\alpha_{\Phi}$ gives rise a Subriemannian Structure in the Lie Algebroid E. This Subriemannian Structure is called Affinor Subriemannian Structure.

Let $D_0 = D, D_1 = [D,D], D_2 = [D_1, D_1]$ and so on. If exists integer s so that $D_s = E$ then we obtain Carnot-Caratheodory Algebroid, but in the common case Subriemannian Metric $d\alpha_{\Phi}$ can't be identified with Carnot-Caratheodory Metric like it done for manyfolds.

To extend an Affinor Subriemannian Structure to the Riemannian Structure in whole Lie Algebroid E we need to define the following useful object:

Definition 4.

A Radical Metric in the Lie Algebroid E is a symmetric 2-form β that satisfies the following two conditions:

- restriction of 2-form β to $\operatorname{rad}\alpha$ is the Riemannian Metric in the vector subbundle $\operatorname{rad}\alpha$;
- for any $\sigma \in D, \tau \in E, \quad \beta(\sigma, \tau) = 0$.

Now, we can define the Riemannian Structure g_Φ in the Lie Algebroid E considering

$$g_\Phi = d\alpha_\Phi + \beta.$$

This Riemannian Structure is called Affinor Metric Structure.

Affinor definition and Affinor Metric Structure construction follow that for any sections $\sigma, \tau \in E$,

$$g_{\Phi}(\Phi\sigma, \tau) = d\alpha(\sigma, \tau).$$

I.e. $d\alpha$ is the fundamental 2-form of Affinor Metric Structure g_{Φ}. Since this fundamental 2-form always is closed, we obtain that Affinor Metric Structure is generalization of a Kähler structure for Lie Algebroids with arbitrary dimension possessed by A regular 1-form. When $E = TM$, if $\operatorname{rad}\alpha = \{0\}$ then $\beta = 0$ and the Affinor Metric Structure precisely is An Almost Kähler Structure over manyfold M. If α is a Contact structure over M then $\operatorname{rank}(\operatorname{rad}\alpha) = 1$ and Affinor Metric Structure g_{Φ} precisely is a Contact Metric Structure over manyfold M.