Regularity of isometries between subRiemannian manifolds

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What is this talk about?

- Every distance-preserving homeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth diffeomorphism, obtained by composition of translations and rotations. In particular $ISO(\mathbb{R}^n, Eucl)$ is a finite dimensional Lie group.

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- In 1939, Myers and Steenrod proved that every distance-preserving homeomorphism (henceforth called isometry) between two Riemannian manifolds is a smooth diffeomorphism.
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  Moreover, the group of Riemannian isometries $ISO(M, g)$ of a manifold (M, g) is a Lie group.
- In this talk I will describe a generalization of these results to the subRiemannian setting, which is due to Capogna, Ottazzi, and myself.

Sketch of the proof in the Riemannian case

After Myers-Steenrod, the Riemannian case has been investigated in a number of papers, for instance Palais (1957), Calabi-Hartman (1970) and Taylor (2006). Taylor's proof is based on harmonic coordinates, here is the simple argument:

Let (M, g_M) be a Riemannian manifold an denote by $d(\cdot, \cdot)$ the corresponding Riemannian distance function. Let $f : M \to M$ be a homeomorphism such that $d(f(x), f(y)) = d(x, y)$.

We want to show that f is smooth.

Sketch of the proof in the Riemannian case

- Since f is Lipschitz then it is a.e. differentiable
  $df : T_x M \to T_{f(x)} M$ is an isometry.
- f maps the $n-$dimensional Hausdorff measure $\mathcal{S}^n_M$ into itself,
  i.e. $\mathcal{S}^n_M(f(A)) = \mathcal{S}^n_M(A)$ for every Borel set $A \subset M$.
  $\mathcal{S}^n_M$ is equal to a constant multiple of the Riemannian volume measure. Hence f maps the Riemannian volume form into itself.
- There exists harmonic coordinates

Sketch of the proof in the Riemannian case

- For any $u \in Lip(M, g)$ one has that for a.e. $x \in M$
  $||\nabla_g u||_{g(f(x))} = ||\nabla_g (u \circ f)||_{g(x)}$
- For any $u \in Lip(M, g)$ one has that

$$\int_M ||\nabla_g (u \circ f)||^2_{g(f(x))} d\mathcal{S}^n_M(x) = \int_M ||\nabla_g u||^2_{g(x)} d\mathcal{S}^n_M(x)$$

  In particular harmonic functions are mapped into harmonic functions.
- If $x_1, ..., x_n$ is a system of harmonic coordinates then $x_i \circ f$ is also harmonic, hence smooth for all $i = 1, ..., n$.

□

SubRiemannian manifolds

Skip

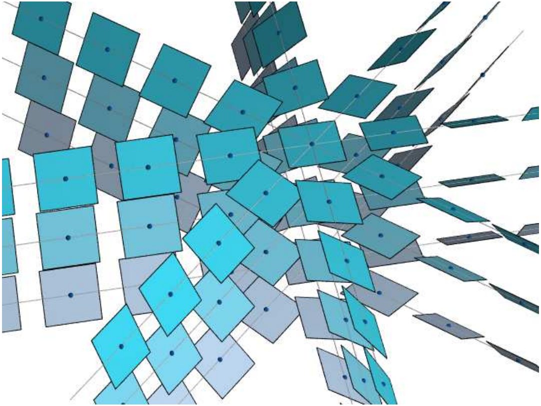
A subRiemannian manifold is a triplet (M, Δ, g) where

- M is a connected smooth manifold,
- Δ is a subbundle of the tangent bundle TM,
- g is a positive-definite smooth bilinear form defined on Δ.

We assume that Δ bracket generates TM: iteratively set $\Delta^1 := \Delta$, and $\Delta^{i+1} := \Delta^i + [\Delta^i, \Delta]$ for $i \in \mathbb{N}$; the bracket generating condition (also called *Hörmander's finite rank hypothesis*) is expressed by the existence of $m \in \mathbb{N}$ such that, for all $p \in M$,

$$\Delta_p^m = T_p M.$$

SubRiemannian manifolds



SubRiemannian manifolds

——— Skip ———

Analogously to the Riemannian setting, one can endow (M, Δ, g) with a metric space structure by defining the *Carnot-Caratheodory* (CC) distance: For any pair $x, y \in M$ set

$$d(x,y) = \inf\{\mathrm{Length}_g(\gamma) \ : \ \gamma \in C^\infty([0,1]; M) \text{ with endpoints } x, y \text{ such that } \dot\gamma \in \Delta_\gamma\}.$$

By Chow-Rashevsky Theorem, (M, d) is a metric space: such a distance is always finite and induces on M the original topology.

SubRiemannian manifolds

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Curves whose velocity vector lies in Δ are called *horizontal*. A horizontal curve is a *geodesic* if it is locally distance minimizing. A geodesic is *normal* if it satisfies the subRiemannian analogue of the geodesic equation. One of the striking features of subRiemannian

SubRiemannian manifolds

Definition

A subRiemannian manifold (M, Δ, g) is equiregular if, for all $i \in \mathbb{N}$, the dimension of Δ_p^i is constant in $p \in M$.
In other words,

$$\Delta^1 \subseteq \Delta^2 \subseteq \cdots \subseteq \Delta^m = TM$$

is a flag of subbundles.

As a result of Mitchell, the Hausdorff dimension of (M, d) coincides with the so-called homogenous dimension

$$Q := \sum_{i=1}^{m} i \, [\dim(\Delta_p^i) - \dim(\Delta_p^{i-1})]. \tag{1}$$

SubRiemannian manifolds

Any arbitrary subRiemannian structure is equiregular in an open dense subset.

For this talk:
subRiemannian manifold = equiregular smooth subRiemannian manifold (with bracket generating subbundle)

SubRiemannian manifolds: Example

——— Skip ———

Heisenberg group: This is a nilpotent Lie group whose underlying manifold is $M = \mathbb{R}^3$. The horizontal distribution is given by

$$\Delta = span\left\{(1, 0, -y);\ (0, 1, x)\right\}$$

and the metric g is defined so that the frame above is orthonormal.

This is a subRiemannian manifold, with homogeneous dimension $Q = 4$. The geodesics are all smooth and normal.

This is the standard example for the class of Carnot groups i.e., nilpotent stratified Lie groups endowed with a subRiemanian metric.

Carnot groups

- Carnot groups are exactly the metric tangent spaces (à la Gromov) of subRiemannian manifolds.

Carnot groups

- Carnot groups are exactly the metric tangent spaces (à la Gromov) of subRiemannian manifolds.
- The Euclidean space $\mathbb{R}^n$ is the metric tangent at any point of any Riemannian n-manifold.
- Carnot groups are particular Lie groups (nilpotent and admitting a one-parameter family of automorphisms δ_λ) equipped with a left-invariant subRiemannian structure with respect to the subbundle $\mathrm{Ker}(\delta_\lambda - \lambda)$.

Main results: SubRiemannian setting

Theorem (L.Capogna & E.L.D.)

Isometries between subRiemannian manifolds are smooth.

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Theorem (L.Capogna & E.L.D.)

Isometries between subRiemannian manifolds are smooth.

Theorem (Hamenstädt, Kishimoto, E.L.D. & Ottazzi)

Isometries between open sets in Carnot groups are affine maps, i.e., composition of translations and isomorphisms.

Consequences: SubRiemannian setting

Corollary

If M is a subRiemannian manifold, then $\mathrm{Isom}(M)$ admits a structure of finite-dimensional Lie group.

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Corollary

Any isometry F between subRiemannian manifolds is determined by the value $F(p)$ at an arbitrary point p and the horizontal differential $dF|_{\Delta_p}$.

Consequences: SubRiemannian setting

Corollary

If M is a subRiemannian manifold, then for all compact subgroup $K < \mathrm{Isom}(M)$, there exists a Riemannian extension g_K on M such that $K < \mathrm{Isom}(M, g_K)$.

Consequences: SubRiemannian setting

Corollary

If M is a subRiemannian manifold, then for all compact subgroup $K < \mathrm{Isom}(M)$, there exists a Riemannian extension g_K on M such that $K < \mathrm{Isom}(M, g_K)$.

In particular, since the group of isometries that fixes a point $p \in M$, denoted by $\mathrm{Isom}_p(M)$, is compact (by Ascoli-Arzelà) then we have:

Corollary

If M is a subRiemannian manifold, then for all $p \in M$, there exists a Riemannian extension $\hat{g}$ on M such that $\mathrm{Isom}_p(M) < \mathrm{Isom}(M, \hat{g})$.

Previous contributions

- Hamenstädt, 1990 proved the theorem with the hypothesis that there are only normal geodesics.
- Kishimoto, 2003 sketched an argument to adapt Hamenstädt proof to show that every global isometry between Carnot groups is smooth and then affine.
- Ottazzi and myself gave a different argument for isometries between open subsets of a Carnot group. The regularity part of our proof follows from the work of Capogna and Cowling and is based on regularity for nonlinear degenerate elliptic PDE.

Difficulties #1: Differentiation

The Tangent space is not obtained by freezing coefficients: The *tangent cone* of the metric space (M, d) at a point $p \in M$ is the Gromov-Hausdorff limit $\mathcal{N}_p(M) := \lim_{t \to 0}(M, d/t, p)$. In view of Rothschild-Stein and of Mitchell's work the metric space $\mathcal{N}_p(M)$ is described by the nilpotent approximation associated to the spaces Δ^i_p. In particular, $\mathcal{N}_p(M)$ is a Carnot group. The analogue of

Rademacher Theorem is a deep result of Margulis-Mostow, it does not yield a linear map $df : T_pM \to T_{f(p)}M$ but a Lie group isomorphism between the nilpotent tangent cones.

Difficulties #2: Hausdorff measure misbehaves

Given any C^∞ volume form vol_M on M, we have a C^∞ volume form induced by vol_M on the nilpotent approximation $\mathcal{N}_p(M)$, which we denote by $\mathcal{N}_p(\mathrm{vol}_M)$.

Theorem (Agrachev, Barilari, Boscain 2012)

Denote by Q the Hausdorff dimension of M and by $\mathcal{S}^Q_M$ the spherical Hausdorff measure on M. Any C^∞ volume form is related to $\mathcal{S}^Q_M$ by

$$\mathrm{d}\,\mathrm{vol}_M = 2^{-Q}\mathcal{N}_p(\mathrm{vol}_M)(B_{\mathcal{N}_p(M)}(e,1))\mathrm{d}\mathcal{S}^Q_M. \qquad (2)$$

$B_{\mathcal{N}_p(M)}(e,1)$ is the unit ball in the metric space $\mathcal{N}_p(M)$ with center the identity element. Unlike the Riemannian case: the density $p \to \mathcal{N}_p(\mathrm{vol}_M)(B_{\mathcal{N}_p(M)}(e,1))$ is not smooth!

Difficulties #3: No harmonic coordinates (so far)

The construction of (Riemannian) harmonic coordinates is ultimately based on freezing the coefficients of the Laplace Beltrami operator, thus reducing the problem to the usual Laplacian in Euclidean space. Freezing coefficients does not work in this setting, as it would lead to loosing the Hörmander finite-rank condition.

We prove the regularity result in two steps:

Theorem

Let $F : M \to N$ be an isometry between two subRiemannian manifolds. If there exist two C^∞ volume forms vol_M and vol_N such that $F_* \mathrm{vol}_M = \mathrm{vol}_N$, then F is a C^∞ diffeomorphism.

A C^∞ *volume form* on an n-dimensional manifold is the measure associated to a C^∞ nonvanishing n-form on the manifold. When $F : M \to N$ is a continuous map, the measure $F_* \mathrm{vol}_M$ is defined by $F_* \mathrm{vol}_M(A) := \mathrm{vol}_M(F^{-1}(A))$, for all measurable sets $A \subseteq N$.

This is a strong hypothesis!

Sketch of the proof in the subRiemannian case

The proof uses an approach informed by the theory of analysis in metric measured spaces. We rely on the concept of upper gradient of a function $u : (X, d) \to \mathbb{R}$ defined as a Borel function $\rho : X \to \mathbb{R}$ such that for any $x, y \in X$ and any $1-$Lipschitz curve γ joining x to y, one has

$$|u(x) - u(y)| \leq \int_\gamma \rho \, \mathrm{d}s$$

The minimal upper gradient is the smallest locally integrable g such that the inequality holds.

Sketch of the proof in the subRiemannian case

Fact 1 Minimal upper gradients are invariant by measure-preserving isometries (trivial)
Fact 2 The minimal upper gradient of a Lipschitz function $u : M \to \mathbb{R}$ on a subRiemanian manifold (M, Δ, g) is $||\nabla_H u||_g$ where ∇_H is the gradient of the function along the fiber $\Delta_p \subset T_pM$. (Hajlasz-Koskela 2003)

Sketch of the proof in the subRiemannian case

Using Fact 1 and 2 one can easily prove that
If $u \in \mathrm{Lip}_{\mathrm{loc}}(N)$ and $g \in L^2(M)$ solve

$$\int_M \langle \nabla_{\mathrm{H}} u, \nabla_{\mathrm{H}} v \rangle \, \mathrm{d}\,\mathrm{vol}_M = \int_M g v \, \mathrm{d}\,\mathrm{vol}_M, \qquad \forall v \in \mathrm{Lip}_c(M),$$

then the functions $\widetilde{u} := u \circ F$ and $\widetilde{g} := g \circ F$ solve

$$\int_M \langle \nabla_{\mathrm{H}} \widetilde{u}, \nabla_{\mathrm{H}} v \rangle \, \mathrm{d}\,\mathrm{vol}_M = \int_M \widetilde{g} v \, \mathrm{d}\,\mathrm{vol}_M, \qquad \forall v \in \mathrm{Lip}_c(M).$$

Sketch of the proof in the subRiemannian case

Theorem (Rothschild-Stein)

Let $X_0, X_1, ..., X_r$ be a system of smooth vector fields in $\mathbb{R}^n$ satisfying Hörmander's finite rank hypothesis. Let

$$\mathcal{L} := \sum_{i=1}^{r} X_i^2 + X_0 \tag{3}$$

and consider a distributional solution to the equation $\mathcal{L}u = f$ in $\mathbb{R}^n$. For every $k \in \mathbb{N} \cup \{0\}$ and $1 < p < \infty$, if $f \in W_{\mathrm{H}}^{k,p}(\mathbb{R}^n, \mathcal{L}^n)$ then $u \in W_{\mathrm{H,loc}}^{k+2,p}(\mathbb{R}^n, \mathcal{L}^n)$.

Sketch of the proof in the subRiemannian case

For any $p \in M$ let us consider now any set of coordinates

$$x_1, \ldots, x_n$$

around $f(p)$. These smooth functions solve for some smooth functions g_i the linear degenerate elliptic PDE

$$\int_M \langle \nabla_{\mathrm{H}} x_i, \nabla_{\mathrm{H}} v \rangle \, \mathrm{d}\,\mathrm{vol}_M = \int_M g_i v \, \mathrm{d}\,\mathrm{vol}_M, \qquad \forall v \in \mathrm{Lip}_c(M), \; i = 1, \ldots, n$$

In view of the invariance by F from the previous slide one has that for all $i = 1, \ldots, n$

$$\int_M \langle \nabla_{\mathrm{H}} F_i, \nabla_{\mathrm{H}} v \rangle \, \mathrm{d}\,\mathrm{vol}_M = \int_M (g_i \circ f) v \, \mathrm{d}\,\mathrm{vol}_M, \qquad \forall v \in \mathrm{Lip}_c(M).$$

where we have set $F_i := x_i \circ F$. Since $g \circ F$ is in L^p_{loc} for all $1 < p < \infty$ then Rotschild-Stein L^p estimates yield $F \in W^{1,p}_{loc}$. The smoothness of F now follows from a bootstrap argument.

Sketch of the proof in the subRiemannian case: Popp's measure

We consider a canonical smooth volume form that is defined only in terms of the subRiemannian structure, the so-called Popp's measure and show the following result.

Theorem

Let $F : M \to N$ be an isometry between subRiemannian manifolds. If vol_M and vol_N are the Popp's measures on M and N, respectively, then $F_* \mathrm{vol}_M = \mathrm{vol}_N$.

The only thing we need to know about Popp's measures vol_M is that their lift to the tangent cone is itself (by left invariance) the popp measure of the tangent cone, i.e.

$$\mathcal{N}_p(\mathrm{vol}_M) = \mathrm{vol}_{\mathcal{N}_p(M)} .$$

Sketch of the proof in the subRiemannian case: Popp's measure

Next: Passing to tangents, there exists a (not a priori unique) isometry $G_p : \mathcal{N}_p(M) \to \mathcal{N}_{F(p)}(M)$, fixing the identity.
In particular, the unit balls of the tangents are the same, i.e.,

$$G_p(B_{\mathcal{N}_p(M)}(e,1)) = B_{\mathcal{N}_{F(p)}(M)}(e,1), \tag{4}$$

and the Popp's measures are preserved:

$$(G_p)_* \operatorname{vol}_{\mathcal{N}_p(M)} = \operatorname{vol}_{\mathcal{N}_{F(p)}(M)},$$

since on Carnot groups Popp's measures are Haar measures.

Sketch of the proof in the subRiemannian case: Popp's measure

$$
\begin{aligned}
F_* \operatorname{vol}_M(A) &= \operatorname{vol}_M(F^{-1}(A)) \\
&= \frac{1}{2^Q} \int_{F^{-1}(A)} \mathcal{N}_p(\operatorname{vol}_M)(B_{\mathcal{N}_p(M)}(e,1)) \, \mathrm{d}\mathcal{S}^Q_M(p) \\
&= \frac{1}{2^Q} \int_{F^{-1}(A)} \mathcal{N}_p(\operatorname{vol}_M) \left(G_p^{-1}(B_{\mathcal{N}_{F(p)}(N)}(e,1)) \right) \mathrm{d}\mathcal{S}^Q_M(p) \\
&= \frac{1}{2^Q} \int_{F^{-1}(A)} (G_p)_* \mathcal{N}_p(\operatorname{vol}_M) \left(B_{\mathcal{N}_{F(p)}(N)}(e,1) \right) \mathrm{d}\mathcal{S}^Q_M(p) \\
&= \frac{1}{2^Q} \int_{F^{-1}(A)} (G_p)_* \operatorname{vol}_{\mathcal{N}_p(M)} \left(B_{\mathcal{N}_{F(p)}(M)}(e,1) \right) \mathrm{d}\mathcal{S}^Q_M(p)
\end{aligned}
$$

Sketch of the proof in the subRiemannian case: Popp's measure

$$= \frac{1}{2^Q} \int_{F^{-1}(A)} \mathrm{vol}_{\mathcal{N}_{F(p)}(M)} \left(B_{\mathcal{N}_{F(p)}(M)}(e,1) \right) \mathrm{d}\mathcal{S}^Q_M(p)$$

...through a chance of variables $q = F(p)$, using invariance of Hausdorff measure by isometry ....

$$= \frac{1}{2^Q} \int_A \mathrm{vol}_{\mathcal{N}_q(M)}(B_{\mathcal{N}_q(M)}(e,1)) \, \mathrm{d}\mathcal{S}^Q_M(q)$$

$$= \frac{1}{2^Q} \int_A \mathcal{N}_q(\mathrm{vol}_M)(B_{\mathcal{N}_q(M)}(e,1)) \, \mathrm{d}\mathcal{S}^Q_M(q)$$

$$= \mathrm{vol}_M(A).$$

This concludes the proof.

Popp's volume for subRiemannian manifolds

Since (M, Δ, g) is equiregular, we have the flag of subbundles:

$$\Delta^1 \subseteq \Delta^2 \subseteq \cdots \subseteq \Delta^m = TM.$$

Fix $q \in M$. Let

$$\mathrm{gr}_q(\Delta) := \Delta_q \oplus \Delta_q^2/\Delta_q \oplus \ldots \oplus \Delta_q^m/\Delta_q^{m-1}$$

$\mathrm{gr}_q(\Delta)$ has a natural volume form, which is the canonical volume of an inner product space obtained by wedging the elements an orthonormal dual basis.
Then there is a canonical isomorphism $\Lambda^n(T_pM) \to \Lambda^n \mathrm{gr}_q(\Delta)$.
This defines an element of $(\Lambda^n T_qM)^* \simeq \Lambda^n T_q^* M$, which is defined to be the Popp's volume form computed at q.

The Montgomery-Zippin Theorem

A key ingredient for our structure result is given by the following theorem:

Theorem (Montgomery-Zippin)

Let G be a locally compact effective transformation group of a connected manifold M and let each transformation of G be smooth. Then G does not contain small subgroups and is a Lie group

The Montgomery-Zippin Theorem

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Let G be a locally compact effective transformation group of a connected manifold M and let each transformation of G be smooth. Then G does not contain small subgroups and is a Lie group

This theorem builds upon work of Yamabe, Gleason, Bochner-Montgomery, and many others. It is related to Hilbert's fifth problem.

SubRiemannian vs Riemannian isometries

Since $\mathrm{Isom}(M)$ is locally compact by Ascoli-Arzelà, from the result of Montgomery and Zippin, we have that $\mathrm{Isom}(M)$ is a Lie group. Fix an auxiliary Riemannian extension $\hat{g}$ of g and define

$$\widetilde{g} = \int_H F_* \hat{g} \,\mathrm{d}\mu_H(F),$$

where μ_H is a probability Haar measure on the compact subgroup $H := \mathrm{Isom}_p(M)$. Notice that $\mathcal{G} := \{F_*\hat{g} : F \in H\}$ is a compact set of Riemannian tensors extending the subRiemannian structure on M. Hence, $\widetilde{g}$ is a Riemannian extention and is $\mathrm{Isom}_p(M)$-invariant. In other words, $\mathrm{Isom}_p(M) < \mathrm{Isom}(M, \widetilde{g})$.

First-order expantion

Lemma

Let (M, Δ, g) be a connected subRiemannian manifold. Let $\widetilde{g}$ be a Riemannian extension of g. Assume that a C^∞ map $F : M \to M$ is a Riemannian isometry for $\widetilde{g}$ and a subRiemannian isometry for g and assume that there exists $p \in M$ such that $F(p) = p$ and $dF|_{\Delta_p} = \mathrm{id}_{\Delta_p}$. Then $dF|_{T_pM} = \mathrm{id}_{T_pM}$ and hence $F = \mathrm{id}_M$.

Isometries of Lie groups

- There are (sub)Riemannian Lie groups that are isometric, but not isomorphic.
- There are self-isometries of (sub)Riemannian Lie groups that are not automorphisms.
- Isometries between nilpotent Riemannian Lie groups are isomorphisms.

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- Isometries between nilpotent subRiemannian Lie groups