Construction of Classical Metrics

with Special Holonomies via Geometrical Flows

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Holonomy of Riemannian manifold

Let (M^n, g) be a Riemannian manifold, $p \in M^n$.

$Hol_p(M^n) = \{P_\gamma | \gamma(t), 0 \leq t \leq 1$ — a loop in $M^n, \gamma(0) = \gamma(1) = p\}$,

where P_γ — parallel transport along the curve γ with respect to the Levi-Civita connection.

$$Hol_p(M^n) \subset Iso(T_p M^n) = SO(n).$$

Products and Symmetrical spaces

Theorem (de Rham, 1952)

Let M — be a complete Riemannian manifold and $Hol(M) = G_1 \times G_2$.
Then $M = M_1 \times M_2$, where $Hol(M_1) = G_1$ and $Hol(M_2) = G_2$.

Theorem (E. Cartan, 1926)

Let $M^n = G/H$ be a symmetrical space (G — Lee group generated by all symmetrical reflections that turn over the geodesics pass through point $p \in M$, H is a stabilizer of p).
Then $H = Hol(M)$.

Holonomy and curvature

Theorem (Ambrose-Singer)

Let M be a connected paracompact Riemannian manifold. Then the Lie algebra $\mathfrak{hol}(M)$ of the holonomy $Hol(M)$ is a subalgebra in $so(n)$ generated by the elements $\Omega(X, Y)$, where Ω is a curvature 2-form.

$$\Omega^i_j = \frac{1}{2} R^i_{jkl} \varepsilon^k \wedge \varepsilon^l$$

Fabel, Gorodski and Rumin defined the holonomy group of the sub-Riemannian manifolds of contact type. They used some adapted connection for it and proved the analog of Ambrose-Singer theorem.

Classification

Theorem (Berger, 1955)

Let M^n be a simply connected irreducible non-symmetrical Riemannian manifold. Then one of the following cases holds:

1. $Hol(g) = SO(n)$,
2. $Hol(g) = U(m), \quad n = 2m \geq 4$,
3. $Hol(g) = SU(m), \quad n = 2m \geq 4$,
4. $Hol(g) = Sp(m), \quad n = 4m \geq 8$,
5. $Hol(g) = Sp(m)Sp(1), \quad n = 4m \geq 8$,
6. $Hol(g) = G_2, \quad n = 7$,
7. $Hol(g) = Spin(7), \quad n = 8$.

Cones and 3-Sasakian manifolds

The Riemannian cone over Riemannian manifold (M, ds^2) is

$$C(M) = \mathbb{R}_+ \times M$$

with metric

$$dt^2 + t^2\ ds^2.$$

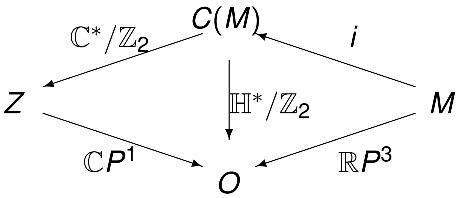
M is 3-Sasakian if there are 3 unit Killing vector fields $\xi_i,\ i = 1, 2, 3$ on M such that $[\xi_i, \xi_j] = 2\epsilon_{ijk}\xi_k$ and that $(1, 1)$-tensor field $\Phi_i = \nabla\xi_i$ satisfies $(\nabla_X\Phi_i)(Y) = g(Y, \xi_i)X - g(X, Y)\xi_i$.

Theorem (Boyer-Galicki)

Let M^n be a complete 3-Sasakian manifold. Then $C(M)$ is hyperkahler, i.e. $Hol(C(M)) = Sp(\frac{n+1}{4})$.

Connections with other geometries

There is a diagram, which called a 'diamond diagram', that connect different geometries:



Deformations of the cone

We suppose that M is a 7-dimensional 3-Sasakian manifold and that quaternionic Kähler orbifold O of M is Kähler. For example, $SU(3)/U(1)$ is a such manifold.

We consider a deformation of the cone metric:

$$g(t) = dt^2 + A_1(t)^2\eta_1^2 + A_2(t)^2\eta_2^2 + A_3(t)^2\eta_3^2$$
$$B(t)^2(\eta_4^2 + \eta_5^2) + C(t)^2(\eta_6^2 + \eta_7^2)$$

and want to find metric g such that $Hol(g) = Spin(7)$.

$$Sp(2) \subset SU(4) \subset Spin(7) \subset SO(8)$$

Condition for $Hol \subseteq Spin(7)$

$$Hol(g) \subset Spin(7) \quad \Leftrightarrow \quad \nabla\Psi = 0 \quad \Leftrightarrow \quad d\Psi = 0 \quad \Leftrightarrow$$

$$\begin{cases} A_1' = \dfrac{(A_2-A_3)^2-A_1^2}{A_2A_3} + \dfrac{A_1^2(B^2+C^2)}{B^2C^2}, \\ A_2' = \dfrac{A_1^2-A_2^2+A_3^2}{A_1A_3} - \dfrac{B^2+C^2-2A_2^2}{BC}, \\ A_3' = \dfrac{A_1^2+A_2^2-A_3^2}{A_1A_2} - \dfrac{B^2+C^2-2A_3^2}{BC}, \\ B' = -\dfrac{CA_1+BA_2+BA_3}{BC} - \dfrac{(C^2-B^2)(A_2+A_3)}{2A_2A_3C}, \\ C' = -\dfrac{BA_1+CA_2+CA_3}{BC} - \dfrac{(B^2-C^2)(A_2+A_3)}{2A_2A_3B}. \end{cases}$$

Resolutions of the cone singularity

Space $\mathcal{M}_1^8$:

(1) $A_1(0) = A_2(0) = A_3(0) = 0, |A_1'(0)| = |A_2'(0)| = |A_3'(0)| = 1$;
(2) $B(0) \neq 0, B'(0) = 0$;
(3) $C(0) \neq 0, C'(0) = 0$;
(4) functions A_1, A_2, A_3, B, C have definite signs on $(0, \infty)$.

Space $\mathcal{M}_2^8$:

(1) $A_1(0) = 0, |A_1'(0)| = 4$;
(2) $A_2(0) = -A_3(0) \neq 0, A_2'(0) = A_3'(0)$,
(3) $B(0) \neq 0, B'(0) = 0$;
(4) $C(0) \neq 0, C'(0) = 0$;
(5) functions A_1, A_2, A_3, B, C have definite signs on $(0, \infty)$.

1-dimensional family for Calabi metrics

Theorem (Bazaikin-M.)

For $0 \leq \alpha < 1$ every metric from the family

$$g_\alpha = \frac{r^4(r^2-\alpha^2)(r^2+\alpha^2)}{r^8-2\alpha^4(r^4-1)-1}dr^2 + \frac{r^8-2\alpha^4(r^4-1)-1}{r^2(r^2-\alpha^2)(r^2+\alpha^2)}\eta_1^2 + r^2(\eta_2^2+\eta_3^2)$$

$$+(r^2+\alpha^2)(\eta_4^2+\eta_5^2) + (r^2-\alpha^2)(\eta_6^2+\eta_7^2),$$

is complete smooth Riemannian $SU(4)$-holonomy metric on the square of the canonical complex line bundle over the space of flags in $\mathbb{C}^3$. Metric g_0 is isometric to the Calabi metric with holonomy $SU(4)$; and metric g_1 is isometric to the Calabi metric with holonomy $Sp(2) \subset SU(4)$ on the $T^*\mathbb{C}P^2$.

Different point of view

Let us now consider the deformation of the cone $C(M)$ as an evolution of the M under some specific geometric flow.

We want to find some reasonable geometric flow on arbitrary 3-Sasakian 7-dimensional manifold $(M, \bar{g}(t))$ such that cone $C(M)$ with metric $dt^2 + \bar{g}(t)$ has special holonomy group.

$$\frac{\partial}{\partial t}\bar{g}(t) = RHS(\bar{g})$$

Dirac flow

Theorem

Let S^3 be a 3-dimensional sphere with conformally round metric

$$\bar{g}(t) = f^2(t)(\eta_1^2 + \eta_2^2 + \eta_3^2)$$

which satisfies the following flow

$$\frac{\partial}{\partial t}\bar{g}(t) = \sqrt{\bar{Ric} - 4K}.$$

Then the cone $C(S^3)$ with metric $dt^2 + \bar{g}(t)$ is isometric to the space with constant curvature K for $K \in \{-1,\ 0,\ +1\}$.

Note that for $K = +1$ at $t = 2\pi$ sphere S^3 is collapsed and the cone $C(S^3)$ with such metric turns out to be a sphere S^4.

Ricci flow for the metric with two parameters

If we consider a metric

$$\bar{g}(t) = A_1^2(t)\eta_1^2 + A_2^2(t)(\eta_2^2 + \eta_3^2).$$

Then a Ricci flow

$$\bar{g}(t) = -2Ric(\bar{g}(t))$$

for this metric will be equivalent to the system

$$\begin{cases} \frac{A_1'}{A_1} \cdot \frac{dt}{d\tau} = -4\frac{A_1^2}{A_2^4}, \\ \frac{A_2'}{A_2} \cdot \frac{dt}{d\tau} = -\frac{4}{A_2^2}(2 - \frac{A_1^2}{A_2^2}). \end{cases}$$

This system can be integrable:

$$-\frac{1}{16}\beta - \frac{\sqrt{2}}{128}c_1 \arctan(\frac{4\sqrt{2}\beta}{\sqrt{c_1^2 - 32\beta^2}}) = t + c_2,$$

where $\alpha = A_1^2 = \frac{1}{8}\beta(\beta' + 16)$ and $\beta = A_2^2$.

Horrible flow

Theorem

Let $S^3/\mathbb{Z}_2$ be a 3-dimensional projective space with metric

$$\bar{g}(t) = A_1^2(t)\eta_1^2 + A_2^2(t)(\eta_2^2 + \eta_3^2)$$

satisfying

$$\frac{\partial}{\partial t}\bar{g}(t) = \sqrt{det(\bar{Ric})}\bar{Ric}^{-1}$$

with $A_1(0) = 0,\ A_1'(0) = 2$ and $A_2(0) \neq 0$. Then the metric $dt^2 + \bar{g}(t)$ is isometric to the Eguchi-Hanson metric.

Thanks for your attention.