The Laplace-Beltrami operator on conic and anti-conic surfaces

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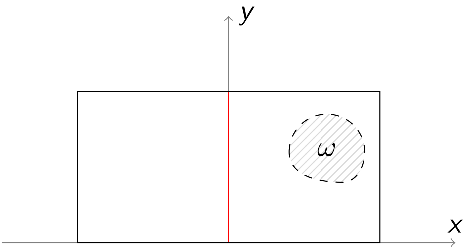
Outline

Grushin Plane

Consider the (non-equiregular) sub-Riemannian control system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = u_1(t)\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{X} + u_2(t)\underbrace{\begin{pmatrix} 0 \\ x \end{pmatrix}}_{Y}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 .$$

Introduced in the context of hypoelliptic operators ($L = X^2 + Y^2 = \partial_x^2 + x^2\partial_y^2$) by Baouendi (1967), Grushin (1970), Franchi-Lanconelli (1984).



It is possible to control the heat evolution associated with L in a square containing the singularity $\mathcal{Z} = \{x = 0\}$, by means of a control localized on one side (Beauchard, Cannarsa and Guglielmi (to appear)).

The vector fields X and Y are linearly independent on $\mathbb{R}^2 \setminus \mathcal{Z}$.

- They define on $\mathbb{R}^2 \setminus \mathcal{Z}$ the Riemannian metric and volume

$$\mathbf{g} = dx^2 + \frac{1}{x^2} dy^2 \qquad dV = \frac{1}{x} dx\, dy.$$

- The singular Laplace-Beltrami operator

$$\mathcal{L}\, u = \operatorname{div} \nabla\, u = \partial_x^2 u - \frac{1}{x} \partial_x u + x^2 \partial_y^2 u.$$

Boscain and Laurent (2013): the heat and the Schrödinger evolutions associated with this operator cannot cross the singularity. Namely,

$$\operatorname{supp} u(0) \subset \{x > 0\} \implies \operatorname{supp} u(t) \subset \{x > 0\} \text{ for any } t > 0,$$

where u is solution of the the heat or of the Schrödinger equations associated with $\mathcal{L}$.

General problem

Given a smooth manifold M endowed with a Riemannian metric and a volume that are degenerate on a singular set $\mathcal{Z} \subset M$, study the properties of the associated Laplace-Beltrami operator.

Examples:

- Almost-Riemannian manifolds
- Non-equiregular sub-Riemannian manifolds
- Manifolds with conic singularities

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Questions

1. The associated Schrödinger or heat diffusion can cross the singularity?
2. Does the singularity absorbs the heat?
3. How does the singularity affect spectral properties?

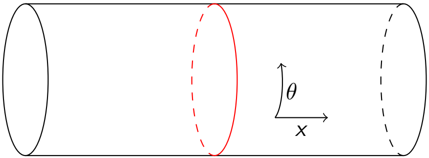
Outline

1. Motivation
2. Diffusions on conic and anti-conic manifolds
3. Spectral analysis of the Grushin cylinder
4. Aharonov-Bohm effect

Outline

Geometrical setting

Consider the manifold $M = \big(\mathbb{R} \setminus \{0\}\big) \times \mathbb{S}^1$.



With the control system

$$\frac{d}{dt}\begin{pmatrix} x \\ \theta \end{pmatrix} = u_1(t)X(x,\theta) + u_2(t)\Theta(x,\theta),$$

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}, \qquad \alpha \in \mathbb{R}$$

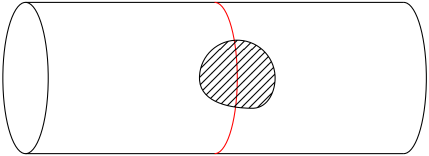
This system defines the Riemannian metric and volume:

$$\mathbf{g}_\alpha = dx^2 + |x|^{-2\alpha}\, d\theta^2, \quad dV = |x|^{-\alpha}\, dx\, dy$$

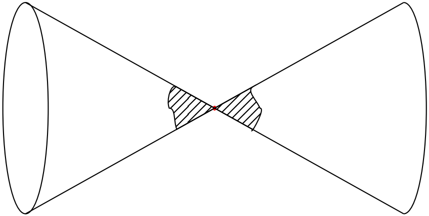
Topological interpretation

$$M_\alpha = \mathbb{R} \times \mathbb{S}^1, \qquad X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}, \qquad \alpha \in \mathbb{R}.$$

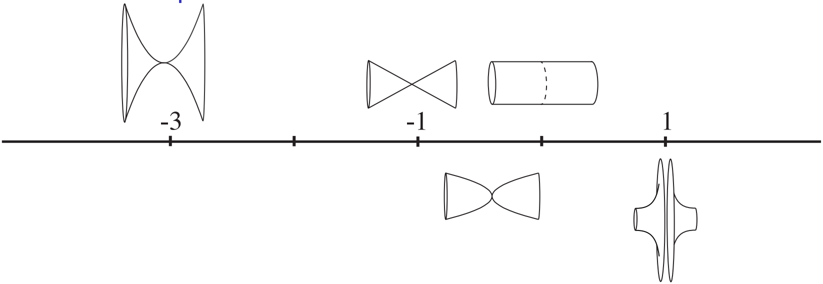
- If $\alpha \geq 0$, the topology is that of a cylinder



- If $\alpha < 0$, the topology is that of a cone.



Metric interpretation



- $\alpha = -1$: Cone
- $\alpha = 0$: Cylinder
- $\alpha = 1$: Grushin plane compactified on one direction
- $\alpha < -1$: conical surface of revolution with profile $x^{-\alpha}$
- $\alpha \in (-1, 0)$: not embeddable in $\mathbb{R}^3$, but we can picture it as a conical surface of rotation
- $\alpha > 0$: not embeddable in $\mathbb{R}^3$, but we think of an "anti-conic" surface of revolution

The Laplace-Beltrami operator

$$\mathcal{L}\, u = \operatorname{div} \nabla\, u = \partial_x^2 - \frac{\alpha}{x}\partial_x + |x|^{2\alpha}\partial_\theta^2.$$

- Schrödinger equation for a free particle:

$$i\frac{\partial}{\partial t}u = -\,\mathcal{L}\, u,$$

- Heat equation:

$$\frac{\partial}{\partial t}u = \mathcal{L}\, u.$$

Questions

1. Is it possible to send a quantum particle from one side to the other of the singularity?
2. Can heat cross the singularity?
3. Is total heat conserved or the singularity absorbs it?

The Laplace-Beltrami operator on $L^2(M, dV)$

Due to the singularity $\mathcal{Z} = \{x = 0\}$, we define $\mathcal{L}$ on $C_c^\infty(M)$.

Give meaning to $\mathcal{L}$ on the singularity $\mathcal{Z}$

$\updownarrow$

Study the self-adjointness of $\mathcal{L}$

Classical results

In any Hilbert space the following equivalences hold,

A self-adjoint operator	$\longleftrightarrow$	e^{-itA} strongly continuous group of unitary transformations
A self-adjoint operator non-positive definite	$\longleftrightarrow$	e^{tA} strongly continuous semi-group

Definition

An operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ (we will always assume $D(A)$ to be dense in $\mathcal{H}$) is *self-adjoint* if

- A is *symmetric* (i.e., if $(Au, v)_{\mathcal{H}} = (u, Av)_{\mathcal{H}}$ for any $u, v \in D(A)$),
- $D(A) = D(A^*)$.

The operator $\mathcal{L}|_{C_c^\infty(M)} : C_c^\infty(M) \to L^2(M, dV)$ is

- **Symmetric:** since if $u \in \mathrm{C}_\mathrm{c}^\infty(M)$, by integration by parts we get

$$(\mathcal{L}\, u, v)_{L^2(M,dV)} = (u, \mathcal{L}\, v)_{L^2(M,dV)} + \Big(\partial_x u\, v - u \partial_x v\Big)\Big|_{0^-}^{0^+}.$$

  for any $v \in L^2(M, dV)$ s.t. $\mathcal{L}\, v \in L^2(M, dV)$.

- **Not self-adjoint:** since

$$D(\mathcal{L}^*) = \{v \in L^2(M, dV) \mid \mathcal{L}\, v \in L^2(M, dV)\}.$$

Self-adjoint extensions of $\mathcal{L}|_{C_c^\infty(M)}$

Definition

An operator A is a self-adjoint extension of $\mathcal{L}|_{C_c^\infty(M)}$ if

$$D\left(\mathcal{L}|_{C_c^\infty(M)}\right) \subset D(A) = D(A^*) \subset D(\mathcal{L}^*)$$
$$Au = \mathcal{L}^* u \quad \text{for any } u \in D(A).$$

The Friedrichs extension $\mathcal{L}_F$ always exists and has domain

$$H_0^2(M, dV) = \{u \in H_0^1(M, dV) \mid \mathcal{L}\, u \in L^2(M, dV)\}$$

Two possibilities:

1. There exists only *one* self-adjoint extension $\longrightarrow \mathcal{L}|_{c_c^\infty(M)}$ is *essentially self-adjoint*.
2. There are *infinitely many* self-adjoint extensions.

Essential self-adjointness and diffusion through the singularity

Observe that,

$$D(\mathcal{L}_F) = H_0^2(M, dV) = H_0^2(\mathbb{R}_+ \times \mathbb{S}^1, dV) \oplus H_0^2(\mathbb{R}_- \times \mathbb{S}^1, dV)$$
$$\Longrightarrow \mathcal{L}_F = \mathcal{L}_F^+ \oplus \mathcal{L}_F^-$$

E.g., when $\alpha = 0$ the Friedrichs extension corresponds to Dirichlet boundary conditions on the singularity.

Proposition

When $\mathcal{L}$ is essentially self-adjoint nothing can cross the singularity.

$\Longrightarrow$ we need to study the essential self-adjointness of $\mathcal{L}|_{C_c^\infty(M)}$.

Fourier decomposition

Through a Fourier decomposition in the θ variable of $\mathrm{L}^2(M, dV)$ the operator $\mathcal{L}$ splits in a family of operators on $H_k = \mathrm{L}^2(\mathbb{R}\setminus\{0\}, |x|^{-\alpha}dx)$.

$$\mathcal{L} = \bigoplus_{k\in\mathbb{Z}} \widehat{\mathcal{L}}_k, \quad \text{s.t.} \quad \widehat{\mathcal{L}}_k = \partial_x^2 - \frac{\alpha}{x}\partial_x - |x|^{2\alpha}k^2.$$

By an unitary transformation U we map $\widehat{\mathcal{L}}_k$ in a Schrödinger operator with Calogero potential x^{-2} on $\mathrm{L}^2(\mathbb{R}\setminus\{0\}, dx)$, that we know how to treat

$$\partial_x^2 - \frac{\alpha}{2}\left(1+\frac{\alpha}{2}\right)\frac{1}{x^2} - k^2|x|^{2\alpha} \quad \text{on } L^2(\mathbb{R}, dx).$$

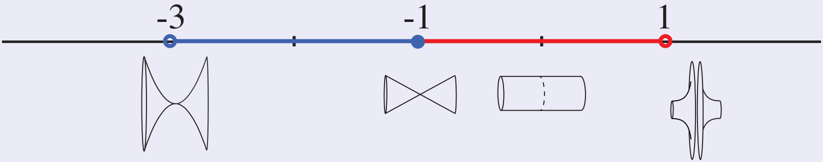
Remark

If $\alpha < -1$ and $k \neq 0$ the potential $k^2|x|^{2\alpha}$ dominates the one in x^2.

Free particle transmission

Theorem

For $\alpha \notin (-3, 1)$ it is not possible to transmit information by means of the Schrödinger equation. On the other hand, for $\alpha \in (-3, -1]$ it is possible to transmit only the average value of the function, while for $\alpha \in (-1, 1)$ we can obtain full communication through the singularity.



- For $\alpha \in (-1, 0)$ the manifold has the topology of a cone, but we can transmit "rotational information".
- Morancey (preprint, 2013) showed that, in a slightly different setting, approximate controllability does not hold for the self-adjoint extension realizing the full communication for $\alpha \in (0, 1)$, with control localized on one side.

Transmission of heat

To accept a self-adjoint extension of $\mathcal{L}|_{C_c^\infty(M)}$, we need an additional condition.

Definition

The self-adjoint operator A on $\mathrm{L}^2(M, dV)$ is *Markovian* if it is non-positive definite and

$$u \in L^2(M, dV) \text{ s.t. } 0 \leq u \leq 1\, a.e. \Longrightarrow 0 \leq e^{tA} u \leq 1\, a.e.$$

- This property can be seen as a physical admissibility condition.
- Every Markov operator (with an additional regularity property, always satisfied in our cases) is the generator of a left-continuous Markov process, Fukushima (1970).

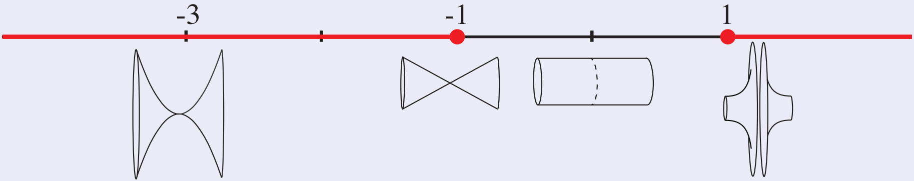
Definition

If the operator $\mathcal{L}|_{C_c^\infty(M)}$ admits only one Markovian extension it is *Markov unique*.

Markov unicity and heat transmission

Theorem

- *If $\alpha \leq -1$ or $\alpha \geq 1$, then $\mathcal{L}|_{C_c^\infty(M)}$ is Markov unique;*
- *If $\alpha \in (-1,1)$, then there are infinitely many Markovian extensions.*



This theorem shows that heat transmission is possible only for $\alpha \in (-1,1)$. We let the *bridging extension* $\mathcal{L}_B$ to be the Markovian extension realizing the maximal communication:

$$D(\mathcal{L}_B) = \{u \in H^2(\bar{M}, dV) \mid u(0^+,\cdot) = u(0^-,\cdot), \lim_{x\to 0^+} |x|^{-\alpha}\partial_x u(x,\cdot) = \lim_{x\to 0^-} |x|^{-\alpha}\partial_x u(x,\cdot) \text{ for a.e. } \theta \in \mathbb{S}^1\}$$

The conservation of heat

The Markov property allows to extend e^{tA} from $L^2(M, dV)$ to $L^\infty(M, dV)$.

Definition

The Markov operator A is *stochastically complete* if

$$e^{tA}1 = 1 \qquad \text{for any } t \geq 0.$$

Question 3

The Markov extensions of $\mathcal{L}|_{C_c^\infty(M)}$ are stochastically complete?

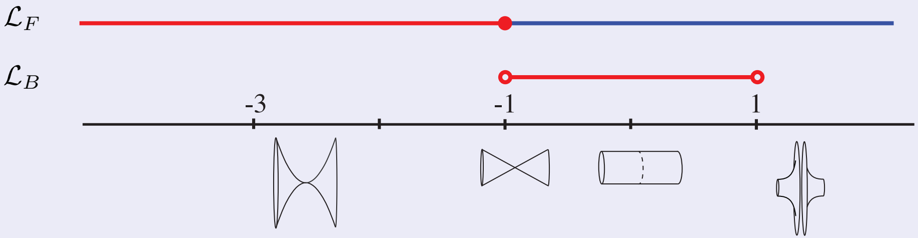
In complete Riemannian manifolds (where the Laplace-Beltrami operator is always essentially self-adjoint) this property is related with

- uniqueness of bounded solutions of the Cauchy problem (Khas'minskii 1960),
- volume explosion at infinity (Grygorian 1985).

Stochastic completeness of $\mathcal{L}_F$, $\mathcal{L}_B$ and $\mathcal{L}_N$

Theorem

- *The Friedrichs extension $\mathcal{L}_F$ is stochastically complete if $\alpha \le -1$ and incomplete if $\alpha > -1$.*
- *If $\alpha \in (-1, 1)$, then the bridging extensions $\mathcal{L}_B$ is stochastically complete .*



Techniques

- The proofs of these facts are based on potential theory and in particular on the theory of Dirichlet forms.
- The fact that $\mathcal{L}|_{C_c^\infty(M)}$ be Markov unique is equivalent to $H^1(M_\alpha, dV) = H_0^1(M_\alpha, dV)$.
- For $\alpha < 1$, to prove the stochastic completeness in 0 of a Markovian extension, it is enough to show that $1\eta \in H^1(M_\alpha, dV)$ and $A1 = 0$, where $\eta \in C_c^\infty(M)$ is a cut-off function s.t. $\eta \equiv 1$ near 0.

Outline

The Grushin cylinder

Recall that the Grushin cylinder is the control system defined on $M = \big(\mathbb{R} \setminus \{0\}\big) \times \mathbb{S}^1$ by the vector fields

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} 0 \\ x \end{pmatrix}, \qquad \alpha \in \mathbb{R}$$

The associated Laplace-Beltrami operator is

$$\mathcal{L}\, u = \partial_x^2 - \frac{1}{x}\partial_x + x^2 \partial_\theta^2.$$

It is essentially self-adjoint on the two sides of the singularity, thus we will consider it as defined on $L^2(\mathbb{R}_+ \times \mathbb{S}^1)$.

Spectral analysis

Proposition

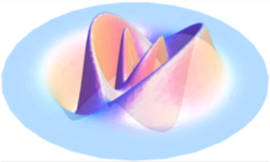
The operator $-\mathcal{L}$ has absolutely continuous spectrum $\sigma(-\mathcal{L}) = [0, +\infty)$ with embedded discrete spectrum

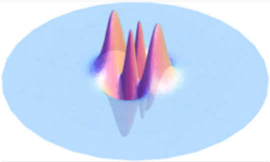
$$\sigma_{\mathrm{d}} = \{\lambda_{n,k} = 4|k|n \mid n \in \mathbb{N}, k \in \mathbb{Z} \setminus \{0\}\}$$

The corresponding eigenfunctions are given by

$$\psi_{n,k} = e^{ik\theta} W_{n,\frac{1}{2}}(|k|x^2),$$

where $W_{\nu,\mu}$ is the Whittaker function of parameters ν and μ.





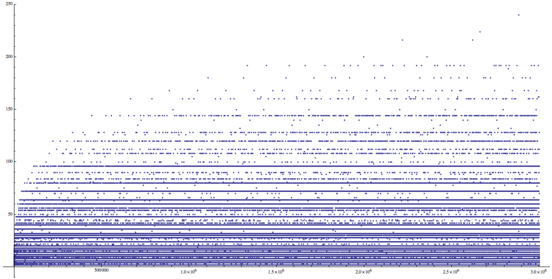
Degeneration of the eigenvalues

Theorem

The degeneracy of an eigenvalue $\lambda \in \sigma_{\mathrm{d}}(-\mathcal{L})$ *is exactly*

$$\begin{cases} 2d(\lambda/4) & \text{if } \lambda/4 \text{ is odd,} \\ 2d(\lambda/4) - 2 & \text{if } \lambda/4 \text{ is even.} \end{cases} \tag{1}$$

Here, $d(n)$ *denotes the number of divisors of the integer* n*.*



Weyl Law

The counting function is defined as

$$N(E) = \{\lambda \leq E \mid \lambda \in \sigma_{\mathrm{d}}(-\mathcal{L})\}, \qquad E > 0.$$

Corollary

The Weyl Law as $E \to +\infty$ is

$$N(E) = \frac{E}{2} \log E + (\gamma - 2 \log 2) \frac{E}{2} + O(1),$$

where γ is the Euler-Mascheroni constant.

Remark

The Weyl Law for a compact 2D Riemannian manifold is $N(E) \sim E$.

Outline

The magnetic Laplace-Beltrami operator

- A magnetic field on a Riemannian manifold M is an exact real valued 2-form B.
- The vector potential $A \in \text{Vec}(M)$ is a vector field s.t. $B = dA$.
- The magnetic Laplace-Beltrami operator is defined as the operator associated with the bilinear form acting on $u, v \in C_c^\infty(M)$ by

$$\mathcal{E}_A(u,v) = \int_M g((\nabla + A)u, (\nabla + A)v)\, dx.$$

This operator could also be defined as $\mathcal{L}_A\, u = -(d + A)^*(d + A)u$, where d is the de Rham exterior differential.

Aharonov-Bohm effect on $\mathbb{R}^2$

Let $M = \mathbb{R}^2 \setminus \{0\}$ and consider the vector potential (in polar coordinates)

$$A^b = -ib\,\frac{1}{r}\partial_\theta, \qquad b \in \mathbb{R}\,.$$

The Aharonov-Bohm magnetic Laplacian (usually one takes Dirichlet conditions at 0) is, in polar coordinates,

$$\Delta_A = \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\left(\partial_\theta - 2ib\partial_\theta - b^2\right).$$

Let $\frac{d}{dt}\psi_0(t) = \Delta_0\psi_0(t)$ and write $\psi_0 = \psi_0^+ + \psi_0^-$ where $\psi_0^\pm$ are supported on the upper and lower half-planes respectively. Then, up to a global phase, a solution of $\frac{d}{dt}\psi(t) = \Delta_A\psi(t)$ is given by

$$\psi(t,x) = \psi_0^+(t,x)e^{\int_\gamma A\cdot v\,dv} + \psi_0^-(t,x) = \psi_0^+(t,x)e^{-i2\pi b} + \psi_0^-(t,x)$$

We see that when $b \notin \mathbb{Z}$, we see a difference of phase depending on the "strength of the null magnetic field".

Aharonov-Bohm effect on the Grushin cylinder

Consider the Aharonov-Bohm vector potential

$$A^b = -ib\frac{1}{r}\,d\theta$$

The corresponding magnetic Laplace-Beltrami operator is

$$\mathcal{L}^b = \partial_x^2 + -\frac{1}{x}\partial_x + |x|^2(\partial_\theta^2 - 2ib\,\partial_\theta - b^2).$$

Theorem

The operator $-\mathcal{L}^b$ *on* $L^2(M_+)$ *has a non-empty discrete spectral component*

$$\sigma_{\mathrm{d}}(-\mathcal{L}^b) = \left\{\lambda_{n,k}^b := 4n|k-b| \mid n \in \mathbb{N},\ k \in \mathbb{Z}\setminus\{b\}\right\}. \tag{2}$$

When $b \in \mathbb{Z}$ *the operator has in addition absolutely continuous spectrum* $[0,+\infty)$*. When* $b \notin \mathbb{Z}$ *the spectrum has no absolutely continuous part.*

Weyl Law of the magnetic Laplace-Beltrami operator

Corollary

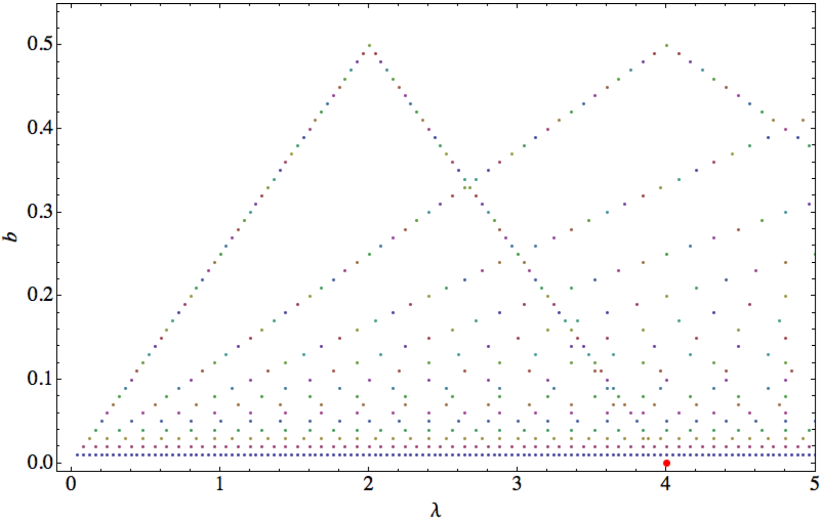
If $b \notin \mathbb{Z}$, let $\kappa \in$ be the closest integer to b. Then, the Weyl law with remainder as $E \to +\infty$ is

$$N(E) = \frac{E}{2}\log(E) + \frac{E}{2}\left(\frac{1}{2|\kappa - b|} + \gamma - 2\log(2) - \frac{\psi(1-|\kappa-b|) + \psi(1+|\kappa-b|)}{2}\right) + O(1),$$

where γ is the Euler-Mascheroni constant and $\psi(x)$ is the digamma function. Here, the $O(1)$ is uniformly bounded with respect to b.

This counting function blows up as $b \to \kappa$. This is due to the fact that a part of the discrete spectrum is degenerating and giving rise to the absolutely continuous spectrum.

Decompactification of the continuous spectrum



Decompactification of the continuous spectrum II

Theorem

Let $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then there exist a sequence of pairs $(b_j, n_j) \in (-\frac{k}{2}, \frac{k}{2}) \times \mathbb{N}$, with $b_j \to k$ and $n_j \to \infty$, such that

$$\psi_{n_j,k}^{b_j}(x,\theta) \to e^{ik\theta} \frac{\sqrt{\lambda}\, x}{2} J_1(\sqrt{\lambda} x)$$

uniformly in x on compact sets, where $J_\nu(z)$ is the Bessel function of the first kind of order ν, that is a generalized eigenfunction for $-\mathcal{L}$ corresponding to the eigenvalue λ.

Degeneration of the eigenvalues

Theorem

Let $d(n)$ denote the number of divisors of n. Then,

- *If $b \in \mathbb{Z}$, the eigenvalues achieve the maximal degeneracy and the multiplicity is exactly*

$$\begin{cases} 2d(\lambda/4), & \text{if } \lambda/4 \text{ is odd,} \\ 2d(\lambda/4) - 2, & \text{if } \lambda/4 \text{ is even,} \end{cases}$$

  (in particular it is bounded below by 2).
- *If $b \in \mathbb{Q}$, the discrete spectrum is degenerate in the following sense: each eigenvalue has multiplicity that is bounded from above by $2d(\lambda/4)$.*
- *If $b \in \mathbb{R} \setminus \mathbb{Q}$, the spectrum is simple.*

Comments

- We remark that, contrarily to the case of the plane, here we do not need to artificially remove the origin. The manifold is already not simply connected.
- Strong effects on the spectrum due to Aharonov-Bohm type magnetic potentials have already been observed, but only in the context of asimptotically hyperbolic manifolds with finite volume.
- Up to our knowledge this is the first time that it is possible to explicitly describe how the continuous spectrum and the corresponding generalized eigenfunctions decompactify.

The Aharonov-Bohm effect on conic and anti-conic surfaces

Considering the same Aharonov-Bohm magnetic potential on the conic/anti-conic manifolds considered before yields the magnetic Laplace-Beltrami operator

$$\mathcal{L}_\alpha^b = \partial_x^2 - \frac{\alpha}{x}\partial_x + |x|^{2\alpha}\left(\partial_\theta - 2ib\partial_\theta - b^2\right)$$

Through the Fourier decomposition the operator on each H_k is

$$\widehat{\mathcal{L}}_{\alpha,k}^b = \partial_x^2 - \frac{\alpha}{x}\partial_x - |x|^{2\alpha}\left(b-k\right)^2$$

Theorem (A-B effect on self-adjointness)

If $\alpha > -1$ the Aharonov-Bohm magnetic potential has no effect on the self-adjointness. On the other hand, if $\alpha \leq -1$, then

1. *if $b \notin \mathbb{Z}$ the operator $\mathcal{L}_\alpha^b$ is essentially self-adjoint;*
2. *if $b \in \mathbb{Z}$ only the $k = b$ component is not essentially self-adjoint.*

THANK YOU FOR YOUR ATTENTION