Comparison Theorems in sub-Riemannian geometry

Luca Rizzi

International School of Advanced Studies (SISSA, Trieste)

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Outline

1. Comparison Theorems in Riemannian geometry
2. Sub-Riemannian Jacobi equation
3. Sub-Riemannian comparison theorems for conjugate points
4. Applications to 3D unimodular Lie groups

Joint work with D. Barilari (Paris 7), preprint available on arXiv

Motivation

M complete Riemannian manifold, $\dim M = m$

- Sectional curvature: $\mathrm{Sec}(v,w) = \mathrm{Rm}(v,w,w,v)$
- Ricci curvature: $\mathrm{Ric}(v) = \sum_{i=1}^{n} \mathrm{Sec}(f_i, v) = \sum_{i=1}^{n} \mathrm{Rm}(f_i, v, v, f_i)$

Theorem

Let γ a unit speed geodesic. If for all $v \in T_{\gamma(t)}M$, $\mathrm{Sec}(\dot\gamma(t), v) \geq \kappa > 0$ then $\gamma(t)$ has a conjugate point at time $t_c(\gamma) \leq \pi/\sqrt{\kappa}$.

Theorem (Bonnet)

If $\forall x \in M$, $\mathrm{Sec}_x \geq \kappa > 0$, then M has diameter not greater than $\pi/\sqrt{\kappa}$. Moreover M is compact, with finite fundamental group.

Motivation

Comparison between a *local property* on M w.r.t. a *model space*:

- local property = sectional curvature, Ricci curvature
- model spaces = space forms

We want to expand these ideas to sub-Riemannian geometry. Ingredients:

- Good definition of curvature
- Sub-Riemannian model spaces

We want to do this for any sub-Riemannian structure

Jacobi fields revisited

- $\lambda \in T^*M$ initial covector
- $\vec{H} \in \mathrm{Vec}(T^*M)$ Hamiltonian vector field
- $\lambda(t) = e^{t\vec{H}}(\lambda)$ strongly normal extremal

For any $\xi \in T_\lambda(T^*M)$, we define the vector field along $\lambda(t)$:

$$X_\xi(t) := e_*^{t\vec{H}}\xi \in T_\lambda(T^*M)$$

Definition

The $2n$-dimensional vector space

$$\{X_\xi(t) | \, \xi \in T_\lambda(T^*M)\} \subset \mathrm{Vec}(\lambda(t))$$

is the *space of Jacobi fields along the extremal* $\lambda(t)$

No connection, no curvature needed.

Jacobi fields revisited (2)

Isomorphism with the space of Jacobi fields along the geodesic

$$\tau \mapsto X_\xi(\tau) = e_*^{\tau\vec{H}}\xi\text{: Jacobi field along the extremal } \lambda(t)$$

$$\Updownarrow \ \pi_* \ \Updownarrow$$

$$\tau \mapsto \pi_* X_\xi(\tau) = \pi_* \circ e_*^{\tau\vec{H}}\xi\text{: Jacobi field along the geodesic } \gamma(t) = \pi \circ \lambda(t)$$

- $\mathrm{ver}_\lambda := \ker \pi_*|_\lambda \subset T_\lambda(T^*M)$ is the *vertical* subspace
- $\mathcal{L}(t) := e_*^{t\vec{H}}\mathrm{ver}_\lambda \subset T_{\lambda(t)}(T^*M)$ (Jacobi fields vanishing at the initial point)

Definition (First conjugate time)

The first conjugate time is the smallest $t > 0$ such that $\mathcal{L}(t) \cap \mathrm{ver}_{\lambda(t)} \neq \{0\}$

- Dimension of the intersection $=$ multiplicity of the conjugate point

Jacobi fields revisited (3)

Natural symplectic setting:

- for all $\lambda \in T^*M$, $T_\lambda(T^*M)$ is naturally a symplectic space
- the flow of $\vec{H}$ is a one-parameter family of symplectomorphisms
- $\mathcal{L}(t) = e_*^{t\vec{H}}\mathrm{ver}_\lambda \subset T_\lambda(T^*M)$ is a family of Lagrangian subspaces

Derivative of fields along the extremal

For any smooth family $X(t)$ of vector fields along the extremal $\lambda(t)$

$$\dot{X}(t) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} e_*^{-\varepsilon\vec{H}} X(t+\varepsilon) \in T_{\lambda(t)}(T^*M)$$

Lie derivative of X in the direction of $\vec{H}$: no connection is required

Moving frames

Aim: recover Jacobi equation, and generalize it to the sub-Riemannian setting

- $\mathrm{ver}_\lambda := \ker \pi_*|_\lambda \subset T_\lambda(T^*M)$ is the *vertical* subspace

Moving frame along the extremal

A frame along the extremal $\lambda(t)$:

$$E^i_{\lambda(t)}, F^j_{\lambda(t)} \in T_{\lambda(t)}(T^*M), \qquad i,j = 1,\dots,n$$

With the following properties:

- $\mathrm{ver}_{\lambda(t)} = \mathrm{span}\{E^i_{\lambda(t)},\ i = 1,\dots,n\}$
- It is a Darboux frame:

$$\sigma(E^i, E^j) = 0, \qquad \sigma(F^i, F^j) = 0, \qquad \sigma(E^i, F^j) = \delta_{ij}$$

σ is the symplectic form on T^*M

Linearized Hamiltonian

Jacobi field along the extremal $X(t) := e_*^{t\vec{H}}\xi$

$$X(t) = \sum_{i=1}^{n} p_i(t) E^i_{\lambda(t)} + x_i(t) F^i_{\lambda(t)} \in T_{\lambda(t)}(T^*M)$$

Proposition (Hamilton equations for the Jacobi fields)

The field $X(t)$ is associated with a curve $t \mapsto p(t), x(t) \in \mathbb{R}^{2n}$ such that

$$\begin{aligned} \dot{p} &= -A_t^* p - Q_t x \\ \dot{x} &= BB_t^* p + A_t x \end{aligned}$$

for some matrices A_t, B_t, Q_t such that $\operatorname{rank} B_t = k$ and $Q_t = Q_t^*$

Hamilton equations in $\mathbb{R}^{2n}$ for the time-dependent Hamiltonian

$$H(p,x) = \frac{1}{2} p^* BB_t^* p + p^* A_t x + \frac{1}{2} x^* Q_t x$$

Linear Quadratic optimal control problem

Linear Quadratic optimal control problem in $\mathbb{R}^n$

$$\dot{x} = A_t x + B_t u, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$
$$\frac{1}{2}\int_0^T \left(|u|^2 - x^* Q_t x\right) \mathrm{d}t \to \min$$

(PMP) Optimal trajectories are solutions of Hamilton equations with

$$H(p,x) = \frac{1}{2} p^* B B_t^* p + p^* A_t x + \frac{1}{2} x^* Q_t x$$

Linearisation of the Hamiltonian flow along the extremal

Let $\lambda : [0, T] \to T^*M$ normal extremal. There is a 1:1 correspondence between Jacobi fields $X(t) = e_*^{t\vec{H}}\xi$ and extremal trajectories $(x,p) : [0, T] \to \mathbb{R}^{2n}$ of the LQ problem with Hamiltonian

$$H(p,x) = \frac{1}{2} p^* B B_t^* p + p^* A_t x + \frac{1}{2} x^* Q_t x$$

The correspondence depends on the choice of the Darboux moving frame

Canonical frame (Riemannian geometry)

Parallel transported frame f_i along $\gamma(t) \Rightarrow$ Darboux frame along $\lambda(t)$ such that

- $A_t = 0$ (no drift)
- $B_t = \mathbb{1}_n$ (constant)
- $(Q_t)_{ij} = \mathrm{Rm}(f_i, \dot\gamma, \dot\gamma, f_j)$

Linearisation of the Hamiltonian flow (Riemannian case)

There is a (canonical) 1:1 correspondence between Jacobi fields along the extremal and extremal trajectories of the LQ problem with Hamiltonian

$$H(p,x) = \frac{1}{2}|p|^2 + \frac{1}{2}x^* Q_t x$$

Remark: the potential is the "directional curvature" (in the direction of γ)

$$x^* Q_t x = \mathrm{Sec}(\dot\gamma(t), v), \qquad v := \sum_{i=1}^n x_i f_i(t) \in T_{\gamma(t)}M$$

Canonical frame (sub-Riemannian geometry)

- Not clear how to "transport a frame" along the geodesic $\gamma(t)$ (no intrinsic affine connection)
- We look for a canonical frame along the extremal $\lambda(t)$ such that the linearised system is simple

Theorem (Agrachev-Zelenko 2002, Zelenko-Li 2009)

For any ample, equiregular extremal $\lambda(t)$ there exists a canonical moving frame such that

- A, B *have a canonical form (constant matrices)*
- Q_t *has particular algebraic symmetries (equations as simple as possible)*

The canonical form of A, B depend on the extremal!

Microlocal flags and growth vectors

$\Delta \subseteq TM$ bracket generating distribution

$$\Delta^{(1)} := \Delta, \qquad \Delta^{(i+1)} := \Delta^{(i)} + [\Delta, \Delta^{(i)}]$$

The *flag of the distribution*: $\Delta^{(1)} \subset \Delta^{(2)} \subset \ldots \subset T_x M$

The *growth vector of the distribution*: $\mathcal{G} = \{\dim \Delta^{(1)}, \dim \Delta^{(2)}, \ldots\}$

$\Downarrow$ microlocal analogue, along the geodesic γ $\Downarrow$

$T \in \mathsf{Vec}(M)$ any horizontal extension of $\dot{\gamma}$

$$\mathcal{D}_\gamma^{(1)} := \Delta, \qquad \mathcal{D}_\gamma^{(i+1)} := \mathcal{D}_\gamma^{(i)} + [T, \Delta^{(i)}]$$

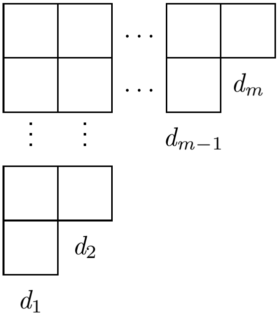
The *flag of the geodesic*: $\mathcal{D}_\gamma^{(1)} \subset \mathcal{D}_\gamma^{(2)} \subset \ldots \subset T_\gamma M$

The *growth vector of the geodesic*: $\mathcal{G}_\gamma = \{\dim \mathcal{D}_\gamma^{(1)}, \dim \mathcal{D}_\gamma^{(2)}, \ldots\}$

- $\mathcal{G}_\gamma$ is "slower" than $\mathcal{G}$ (i.e. $\dim \mathcal{D}_\gamma^{(i)} \leq \dim \Delta^{(i)}$)
- The geodesic is *ample* if $\exists m$ such that $\mathcal{D}_\gamma^{(m)} = T_{x_0} M$
- The geodesic is *equiregular* if $\mathcal{G}_{\gamma(t)}$ does not depend on t

Young diagram of the geodesic

Let $\mathcal{G}_\gamma = \{k_1, k_2, \ldots, k_m\}$ and $d_i := k_i - k_{i-1}$



- $d_1 = k_1 = \dim \Delta$
- Ample geodesics: $\#$ boxes $= \dim M$ (generic condition)
- Length of the rows $\{n_1, \ldots, n_k\} \Longrightarrow$ normal form of the LQ problem

Young diagram of the geodesic

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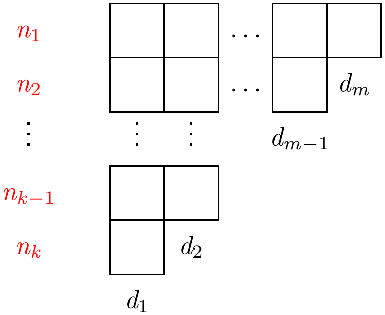

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Normal form of Hamilton equation

Linearisation of the Hamiltonian flow (sub-Riemannian case)

There is a (canonical) 1:1 correspondence between Jacobi fields along the extremal with given Young diagram, with lengths $\{n_1, \ldots, n_k\}$ and extremal trajectories of the LQ problem with Hamiltonian

$$H(p,x) = \frac{1}{2} p^* BB^* p + p^* Ax + \frac{1}{2} x^* Q_t x$$

$$BB^* := \mathrm{diag}\{D_{n_1}, \ldots, D_{n_k}\}, \qquad D_m := \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{O}_{m-1} \end{pmatrix}$$

$$A := \mathrm{diag}\{A_{n_1}, \ldots, A_{n_k}\}, \qquad A_m := \begin{pmatrix} 0 & 0 \\ \mathbb{1}_{m-1} & 0 \end{pmatrix}$$

Sub-Riemannian curvature

Sub-Riemannian directional curvature

In terms of this frame, Q_t defines a quadratic form

$$\mathfrak{R}_{\lambda(t)} : T_{\gamma(t)}M \times T_{\gamma(t)}M \to \mathbb{R}$$

Let $f_i(t) := \pi_* F^i_{\lambda(t)} \in T_{\gamma(t)}M$

$$\mathfrak{R}_{\lambda(t)}(f_i, f_j) := [Q_t]_{ij}$$

Example: Riemannian case

- Every geodesic has the same Young diagram: n rows with length 1

$$A = 0, \qquad BB^* = \mathbb{1}_n$$

- We recover the whole Riemann tensor:

$$\mathfrak{R}_{\lambda(t)}(X, X) = \mathrm{Sec}(X, \dot{\gamma}(t))$$

Sub-Riemannian curvature (comments)

Not completely satisfactory definition of sub-Riemannian curvature $\mathfrak{R}_\lambda$

- Relies on the existence of canonical frame (hard to compute)
- Rather algebraic definition (simplicity of equations)
- Geometrical interpretation?

Riemann's idea: curvature is in the expansion of squared distance

$$d^2(\gamma_v(t), \gamma_w(t)) = 2(1-\cos\theta)t^2\left(1 - \tfrac{1}{3}\kappa\cos^2(\theta/2)t^2\right) + O(t^6), \quad \kappa = \mathrm{Sec}(v,w)$$

Geometrical definition (Agrachev, Barilari, Rizzi 2013)

Restriction of sub-Riemannian curvature $\mathfrak{R}_{\lambda(t)} : \Delta_{\gamma(t)} \times \Delta_{\gamma(t)} \to \mathbb{R}$ can be defined by the expansion of the squared sub-Riemannian distance

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Generalized Jacobi equation and conjugate points

Jacobi fields along the extremal, with given Young diagram $\{n_1, \ldots, n_k\}$

$$X(t) = \sum_{i=1}^{n} p_i(t) E^i_{\lambda(t)} + x_i(t) F^i_{\lambda(t)}$$

Jacobi equation: Hamilton equations for the associated LQ problem

$$(*) \begin{cases} \dot{p} = -A^* p - Q_t x \\ \dot{x} = BB^* p + Ax \end{cases} \xrightarrow[\text{Riemannian case}]{A=0,\, BB^*=\mathbb{1}} \ddot{x} + Q_t x = 0$$

Lemma

t is a conjugate time $\iff \exists$ *solution of* $(*)$ *such that* $x(0) = x(t) = 0$

conjugate points along the extremal = conjugate time for the LQ problem

Riccati equation

Consider the unique symmetric solution of the Cauchy problem

$$(*) \qquad \dot{V} + A^* V + VA + Q_t + VBB^* V = 0, \qquad \lim_{t \to 0^+} V^{-1} = 0$$

- $(*)$ is the matrix Riccati equation for the associated LQ problem
- Cauchy problem well posed if the geodesic is ample
- Its solution blows up in finite time

Proposition

Let $V(t)$ be defined on its maximal interval $I \subset (0, +\infty)$. Let t_1 be the first conjugate time along the extremal. Then $I = (0, t_1)$

- First conjugate time $\equiv$ "blow up time" of Riccati equation of LQ problem

Riccati comparison: if $Q_t \geq Q_{\mathrm{mod}}$, then $V(t) \geq V_{\mathrm{mod}}(t) \geq 0$. If $V_{\mathrm{mod}}(t)$ blows up at t_1, $V(t)$ blows up no later than t_1.

Microlocal comparison theorem

Riccati equation for the Jacobi fields along the extremal

$$\dot{V} + A^* V + VA + \mathfrak{R}_{\lambda(t)} + VBB^* V = 0$$

Riccati equation for a LQ problem with constant Q

$$\dot{V} + A^* V + VA + Q + VBB^* V = 0$$

Definition

- $\mathrm{LQ}(n_1, \dots, n_k; Q)$ denotes the LQ problem with constant potential
- $t(n_1, \dots, n_k; Q)$ is the first conjugate time of the LQ problem

Theorem (Barilari, Rizzi - 2014)

Let $\lambda(t)$ be an ample, equiregular extremal, with Young diagram with k rows, of length $n_1, \dots, n_k$. Let $\mathfrak{R}_{\lambda(t)} : T_{\lambda(t)}M \times T_{\lambda(t)}M \to \mathbb{R}$ the directional curvature along the extremal.

- $Q_- \geq \mathfrak{R}_{\lambda(t)} \geq Q_+ \quad \Rightarrow \quad t(n_1, \dots, n_k; Q_-) \leq t_{\textit{conj}} \leq t(n_1, \dots, n_k; Q_+)$

Microlocal comparison theorem (2)

Theorem (Sub-Riemannian microlocal comparison)

Let $\lambda(t)$ be an ample, equiregular extremal, with Young diagram with k rows, of length $n_1, \dots, n_k$. Let $\mathfrak{R}_{\lambda(t)} : T_{\lambda(t)}M \times T_{\lambda(t)}M \to \mathbb{R}$ the directional curvature along the extremal.

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- The first conjugate time of a LQ problem gives an estimate for the first conjugate time along the geodesic
- The LQ problem with constant potential is a *model* (i.e. we have equality)
- A priori, $t(n_1, \dots, n_k, Q) \not< +\infty$ (more on this later)
- We can "take out the direction of motion" (dimensional reduction)
- If $\mathfrak{R}_{\lambda(t)} \leq 0$, then $t_{\text{conj}} = +\infty$

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Conjugate points of LQ systems

$\mathrm{LQ}(n_1, \ldots, n_k; Q)$ is an optimal control problem in $\mathbb{R}^n$ with

- k controls
- time-independent, linear drift Ax
- constant potential Q

Question: when $t(n_1, \ldots, n_k; Q) < +\infty$?

Hamiltonian vector field of the LQ problem: $\vec{H}(p, x) = \begin{pmatrix} -A^* & -Q \\ BB^* & A \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}$

Theorem (Agrachev - Rizzi - Silveira 2013)

An LQ optimal control problem has finite conjugate time if and only $\vec{H}$ *has at least one Jordan block of odd size with purely imaginary eigenvalue.*

Example: Riemannian case

- Growth vector: $\mathcal{G} = \{m\} \Longrightarrow$ Invariants: $\{\underbrace{1, 1, \ldots, 1}_{m}\}$
- Assumption: $\kappa_- \geq \mathrm{Sec}(\dot\gamma(t), v) \geq \kappa_+ \Longrightarrow$ Potential: $Q_\pm = \kappa_\pm \mathbb{1}$

Theorem (Riemannian microlocal comparison theorem)

Let γ be a geodesic with $\kappa_- \geq \mathrm{Sec}(\dot\gamma(t), v) \geq \kappa_+$ for all unit $v \in T_{\gamma(t)}M$. Then

$$t(1, \ldots, 1; \kappa_- \mathbb{1}) \leq t_{conj}(\gamma) \leq t(1, \ldots, 1; \kappa_+ \mathbb{1})$$

- $\mathrm{LQ}(1, \ldots, 1; \kappa\mathbb{1})$ is the n-dimensional harmonic oscillator

$$H(p, x) = \frac{1}{2}(p^2 + \kappa x^2), \qquad t_c(1, \ldots, 1; \kappa) = \begin{cases} +\infty & \kappa \leq 0 \\ \frac{\pi}{\sqrt{\kappa}} & \kappa > 0 \end{cases}$$

- The geodesic loses optimality in the interval $[t_-, t_+]$, with $t_\pm = \frac{\pi}{\sqrt{\kappa_\pm}}$

Example: Heisenberg group

- Growth vector: $\mathcal{G} = \{2,3\} \Longrightarrow$ Invariants: $\{2,1\}$
- Geodesic with initial covector λ. Recall that $h_0 := \langle \lambda, Z \rangle$ is constant.

$$\mathfrak{R}_{\lambda(t)} = h_0^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: Q \qquad \text{constant along the extremal!}$$

- $\mathrm{LQ}(2,1;Q)$ is a LQ problem in $\mathbb{R}^3$, with Hamiltonian

$$H(p,x) = \frac{1}{2}p_1^2 + p_2 x_1 + \frac{1}{2}h_0^2 x_1^2 \qquad t(2,1;Q) = \begin{cases} +\infty & h_0 = 0 \\ \frac{2\pi}{|h_0|} & h_0 \neq 0 \end{cases}$$

Theorem (Heisenberg microlocal comparison theorem)

Let γ be a geodesic with initial covector λ, then $t_{conj}(\gamma) = \begin{cases} +\infty & h_0 = 0 \\ \frac{2\pi}{|h_0|} & h_0 \neq 0 \end{cases}$

Contact structures on 3D unimodular Lie Groups

- M is a unimodular, simply connected Lie group, $\dim M = 3$
- 1-form ω is the *contact form*. Distribution: $\Delta = \ker \omega$
- left-invariant sub-Riemannian structure $(\Delta, \langle \cdot | \cdot \rangle)$
- X_1, X_2 left-invariant orthonormal frame for $(\Delta, \langle \cdot | \cdot \rangle)$
- X_0 Reeb vector field: $X_0 \in \ker d\omega$, $\omega(X_0) = 1$
- Normalization $d\omega|_\Delta$ is the area element
- Structural constants: $[X_i, X_j] = \sum_{\ell=0}^{2} c_{ij}^{\ell} X_\ell$

Theorem (Agrachev, Barilari - 2012)

The equivalence classes of isometric contact structures on 3D unimodular Lie groups are classified by two invariants: $\chi \geq 0$, $\kappa \in \mathbb{R}$.

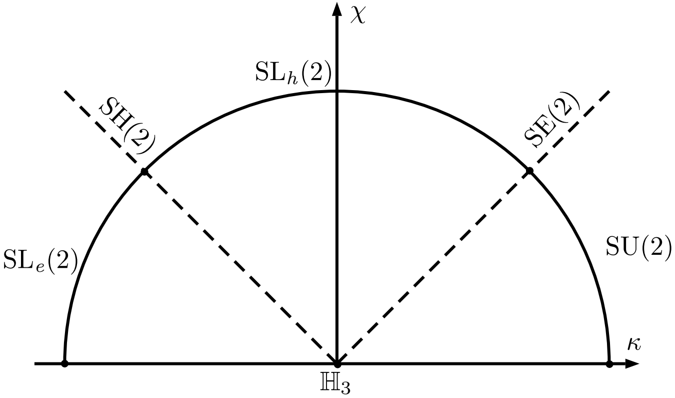
Up to rescaling $\chi^2 + \kappa^2 = 1$.

Contact structures on 3D unimodular Lie Groups

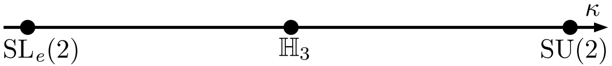
Theorem (Agrachev, Barilari - 2012)

The equivalence classes of isometric contact structures on 3D unimodular Lie groups are classified by two invariants $\chi, \kappa \in \mathbb{R}$.

Up to rescaling and reflections $\chi^2 + \kappa^2 = 1$ and $\chi \geq 0$.



Some known results (case $\chi = 0$)



$h_0 := \langle \lambda, X_0 \rangle$ is always a constant along the extremal

Theorem (Boscain, Rossi - 2008)

Let γ be a geodesic on $\mathrm{SL}(2)$, $\mathrm{SU}(2)$:

- $\mathrm{SL}(2)$ *($\kappa = -1$):* $t_c(\gamma) = \begin{cases} +\infty & h_0^2 \leq 1 \\ \frac{2\pi}{\sqrt{h_0^2 - 1}} & h_0^2 > 1 \end{cases}$
- $\mathrm{SU}(2)$ *($\kappa = 1$):* $t_c(\gamma) = \frac{2\pi}{\sqrt{h_0^2 + 1}}$

Some new results ($\chi > 0$)

Let $\chi > 0$. There exists a left-invariant orthonormal frame X_1, X_2 such that

$$[X_1, X_0] = (\chi + \kappa) X_2,$$
$$[X_2, X_0] = (\chi - \kappa) X_1,$$
$$[X_2, X_1] = X_0$$

Moreover the function $E : T^*M \to \mathbb{R}$ is a constant of the motion

$$E = \frac{h_0^2}{2\chi} + h_2^2, \qquad h_i(\lambda) := \langle \lambda, X_i \rangle$$

Theorem (Barilari, Rizzi - 2014)

Let M be a 3D unimodular Lie group with a left-invariant sub-Riemannian structure, with $\chi > 0$ and $\kappa \in \mathbb{R}$. Then there exists $\overline{E} = \overline{E}(\chi, \kappa)$ such that every length parametrised geodesic γ with $E(\gamma) \geq \overline{E}$ has a finite conjugate time.

This agrees with the results of Sachkov for $\mathrm{SE}(2)$

Main References

[A. Agrachev, D. Barilari, L. Rizzi (2013)]
The curvature: a variational approach (arXiv)

[D. Barilari, L. Rizzi (2014)]
Comparison theorems in sub-Riemannian geometry (arXiv)

[P. Lee, C. Li, I. Zelenko (2009)]
Differential geometry of curves in Lagrange Grassmannians with given Young diagram (Differential Geom. Appl.)

[U. Boscain, F. Rossi (2008)]
Invariant Carnot-Caratheodory metrics on S^3, $\mathrm{SO}(3)$, $\mathrm{SL}(2)$ and lens spaces (SIAM, Journal on Control and Optimization)

[Y. Sachkov (2011)]
Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane (ESAIM: COCV)

- Slides available on my webpage

Averaging - Riemannian setting

Aim: Improve microlocal comparison theorem by averaging

$$\mathrm{Ric}_{\lambda(t)} := \sum_{i=1}^{n} \mathfrak{R}_{\lambda(t)}(f_i, f_i) = \sum_{i=1}^{n} \mathrm{Rm}(f_i, \dot\gamma, \dot\gamma, f_i)$$

Riemannian matrix Riccati equation:

$$\dot V + A^* V + VA + \mathfrak{R}_{\lambda(t)} + VBB^* V = 0, \qquad A = 0,\ BB^* = \mathbb{1}$$

Equation for the trace: $v := \mathrm{trace}\, V/n$

$$\dot v + v^2 + \mathrm{Ric}_{\lambda(t)}/n = 0$$

- Blow up time for $v(t) \Longrightarrow$ blow up time for $V(t) \Longrightarrow$ conjugate time

Theorem (Riemannian average microlocal comparison theorem)

Let M be a Riemannian manifold. Let $\lambda(t)$ be an extremal.

- *If $\mathrm{Ric}_{\lambda(t)} \geq n\kappa$, then $t_{conj} \leq t(1;\kappa)$*

Recall: $\mathrm{LQ}(1;\kappa)$ is the LQ problem in $\mathbb{R}$ with $H(p,x) = \frac{1}{2}(p^2 + \kappa x^2)$

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Averaging - Riemannian setting (2)

Theorem (Riemannian average microlocal comparison theorem)

Let M be a Riemannian manifold. Let $\lambda(t)$ be an extremal.

- *If* $\mathrm{Ric}_{\lambda(t)} \geq n\kappa$, *then* $t_{\mathit{conj}} \leq t(1;\kappa) = \begin{cases} +\infty & \kappa \leq 0 \\ \frac{\pi}{\sqrt{\kappa}} & \kappa > 0 \end{cases}$

- $\mathrm{LQ}(1;\kappa)$ is the LQ problem in $\mathbb{R}$ with $H(p,x) = \frac{1}{2}(p^2 + \kappa x^2)$
- If $\mathrm{Ric}_{\lambda(t)} \geq n\kappa > 0$, the Riemannian geodesic loses optimality at $t = \frac{\pi}{\sqrt{\kappa}}$

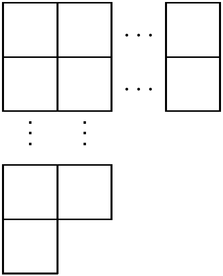
Sub-Riemannian Riccati equation:

$$\dot{V} + A^* V + VA + \mathfrak{R}_{\lambda(t)} + VBB^* V = 0$$

Extra terms, trace not straightforward

Geometrical interpretation of the Young tableau

Flag of the geodesic: $\mathcal{D}_\gamma^{(1)} \subset \mathcal{D}_\gamma^{(2)} \subset \cdots \subset \mathcal{D}_\gamma^{(m)} = T_\gamma M$



- Box $=$ 1-dimensional subspace of $T_\gamma M$
- Shift to the right by acting with Lie derivative $\mathcal{L}_{\dot\gamma}$
- Lengths of rows $\{n_1, \dots, n_k\}$ $=$ how many new directions can be obtained by taking the Lie derivative in the direction of $\dot\gamma$
- Idea: "standard trace" doesn't work: different directions have different "behaviour"

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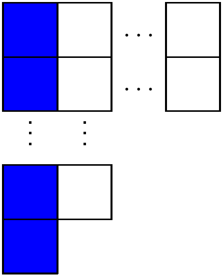

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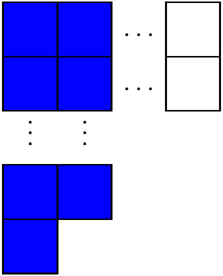

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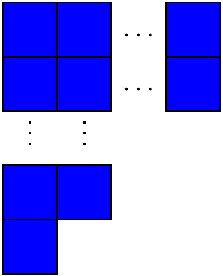

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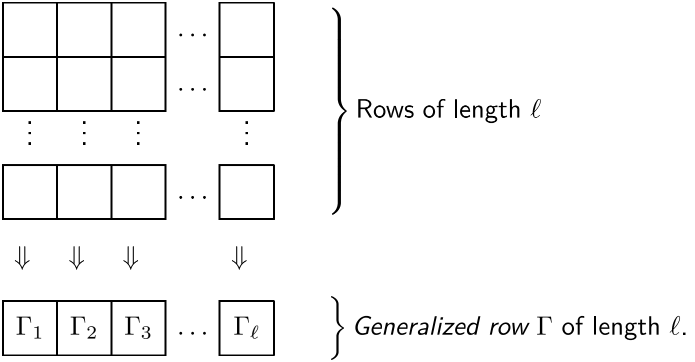
Averaging - sub-Riemannian setting

- Collect boxes (directions) with the same "behaviour" (length)



- Boxes, rows $\Longrightarrow$ *generalized* boxes, rows
- Average of $\mathfrak{R}_{\lambda(t)}$ w.r.t. directions in a gen. box $\Longrightarrow$ Ricci of the gen. box
- Riemannian case: 1 gen. box $\Longrightarrow$ 1 Ricci

Averaging - sub-Riemannian setting (2)

For each gen. row Γ with gen. boxes $\Gamma_1, \dots, \Gamma_\ell$, define the Ricci curvatures

$$\mathrm{Ric}_{\lambda(t)}(\Gamma_i) := \sum_{i \in \Gamma_a} \mathfrak{R}_{\lambda(t)}(f_i, f_i), \qquad a = 1, \dots, \ell$$

Partial trace of matrix Riccati equation, according to Young diagram + matrix Riccati comparison = 1 comparison theorem for each gen. row

Theorem (Sub-Riemannian average microlocal comparison theorem)

Let $\lambda \in T^*M$ be the initial covector of an ample, equiregular geodesic. Assume that, for some gen. row Γ, with gen. boxes $\Gamma_1, \dots, \Gamma_\ell$

$$\frac{\mathrm{Ric}_{\lambda(t)}(\Gamma_a)}{\dim \Gamma_a} \geq \kappa_a, \qquad a = 1, \dots, \ell$$

Then the first conjugate time t_1 on the geodesic is not greater than the first conjugate time of $\mathrm{LQ}(\ell; Q)$ with $Q = \mathrm{diag}\{\kappa_1, \dots, \kappa_\ell\}$

Conjugate points of LQ models

For the models $\mathrm{LQ}(\ell; Q)$ we have $\vec{H}(p,x) = \begin{pmatrix} -A^* & -Q \\ BB^* & A \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 0 \\ \mathbb{1}_{\ell-1} & 0 \end{pmatrix}, \qquad BB^* = \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{O}_\ell \end{pmatrix}, \qquad Q = \mathrm{diag}(\kappa_1, \dots, \kappa_\ell)$$

Algebraic, computable condition in terms of $\kappa_1, \dots, \kappa_\ell$

Riemannian case

One gen. row of length 1. Model: $\mathrm{LQ}(1;\kappa)$ (harmonic oscillator)

- $t_1 < +\infty$ iff $\kappa > 0$

3D contact structure

One gen. row of length 2. Model: $\mathrm{LQ}(2;\kappa_1,\kappa_2)$

- $t_1 < +\infty$ iff $\begin{cases} \kappa_1 > 0, & \kappa_2 > -\kappa_1^2/4 \\ \kappa_1 \leq 0, & \kappa_2 > 0 \end{cases}$

Sub-Riemannian Bonnet-Myers Theorem

- M complete, connected sub-Riemannian manifold
- All the minimizing geodesics have the same growth vector

Theorem (Sub-Riemannian Bonnet-Myers)

Assume that there exists a gen. row Γ*, with gen. boxes* $\Gamma_1, \ldots, \Gamma_\ell$ *and constants* $\kappa_1, \ldots, \kappa_\ell$ *such that, for any length parametrized geodesic, with initial covector* λ

$$\frac{\mathrm{Ric}_\lambda(\Gamma_a)}{\dim \Gamma_a} \geq \kappa_a, \qquad a = 1, \ldots, \ell$$

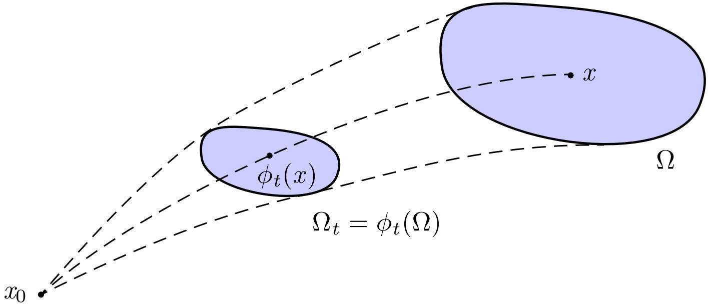
Then, if the polynomial

$$P_{\kappa_1,\ldots,\kappa_\ell}(x) = x^{2\ell} + \sum_{a=0}^{\ell-1} \kappa_{\ell-a} x^{2a} (-1)^{\ell-a-1}$$

has at least one simple imaginary root, the manifold is compact, has finite diameter $\leq t(\ell; \kappa_1, \ldots, \kappa_\ell)$*. Moreover its fundamental group is finite.*

Contraction of volumes

M complete, connected Riemannian manifold, μ smooth volume form
$\Omega \subset M$ measurable, $0 < \mathrm{vol}(\Omega) < \infty$



$$\phi_t(x) := \mathrm{exp}_{x_0}(tv_x) \qquad \text{smooth a.e. on } [0,1] \times M$$

Measure Contraction Property

Theorem (Ohta, 2007)

If $\forall x \in M$, $\mathrm{Ric}_x \geq (n-1)\kappa$, *then for any* x_0 *and measurable set* $0 < vol(\Omega) < \infty$

$$\mathrm{Vol}(\Omega_t) \geq \int_\Omega t \left[\frac{u_\kappa(td(x_0,x))}{u_\kappa(d(x_0,x))} \right]^{n-1} \mu_x, \qquad \forall\, t \in [0,1] \tag{1}$$

where $d : M \times M \to \mathbb{R}$ *is the Riemannian distance and*

$$u_\kappa(s) := \begin{cases} \sin(\sqrt{\kappa} s) & \kappa > 0 \qquad (\textit{sphere}) \\ s & \kappa = 0 \qquad (\mathbb{R}^n) \\ \sinh(\sqrt{-\kappa} s) & \kappa < 0 \qquad (\textit{hyperbolic plane}) \end{cases}$$

Eq. (1) is called Measure Contraction Property: $\mathrm{MCP}(\kappa, n)$