LOCAL AND METRIC GEOMETRY OF NONREGULAR WEIGHTED CARNOT-CARATHÉODORY SPACES

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International Youth Conference “Geometry and Control”,
Moscow, 14-18 April 2014

The setting

⋄ Consider a smooth manifold $\mathbb{M}$, $\dim(\mathbb{M}) = N$.

⋄ Vector fields $X_1, X_2, \ldots, X_q \in \mathrm{Vec}(U)$, $U \subseteq \mathbb{M}$ $\deg(X_i) := d_i$, such that locally

$$[X_i, X_j](v) = \sum_{d_k \leq d_i + d_j} c_{ij}^k(v) X_k(v)$$

("Hermann finite type condition")

$\mathrm{span}\{X_1(v), X_2(v), \ldots X_q(v)\} = T_v\mathbb{M}$ for any $v \in U$

induce a weighted Carnot-Carathéodory space

◇ In terms of filtrations:

• $X_I = [X_{i_1}, [\ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]$, where $I = (i_1, \ldots, i_k)$;
$|I|_h := d_{i_1} + \ldots + d_{i_k}$.

• $H_j = \text{span}\{X_I \mid |I|_h \leq j\}$.

$$H\mathbb{M} = H_{d_1} \subseteq \ldots \subseteq H_{d_q} = T\mathbb{M}$$

$$[H_i, H_j] \subseteq H_{i+j}.$$

Here $[H_i, H_j]$ is the linear span of commutators of the vector field generating H_i and H_j.

$\diamond$ Examples:

• classical sub-Riemannian manifolds (Hörmander condition

$H_{i+1} = [H_1, H_i]$, weights $1, 2, \ldots M$)

• equiregular CC spaces from previous talks ($q = N$; weights $1, 2, \ldots M$)

Model case: $d_1 := 1,\ d_q := M$ (M is the depth of the CC space) $X_i \in C^r$, $r \geq M$.

• A point $u \in \mathbb{M}$ is called regular if $\dim H_k(v) = \text{const}$ in some neighborhood $v \in U(u) \subseteq \mathbb{M}$.

(Remark: in a neighbourhood of a regular point, everything can be reduced to the case of equiregular CC spaces studied in the previous talks)

Otherwise, u is called nonregular. I.e., in any neighbourhood of u the dimensions of H_k may change.

- Examples of nonregular CC spaces:

⋄ Groushin-type planes (or "almost-Riemannian manifolds" related to the PDE $\frac{\partial^2 u}{\partial x^2} + x^{2k}\frac{\partial^2 u}{\partial x^2} = f$)

$\mathbb{M} = \mathbb{R}^2$. $H_1 = \text{span}\{X_1 = \frac{\partial}{\partial x},\ X_2 = x^k \frac{\partial}{\partial y}\}$.

The axis $x = 0$ consists of nonregular points; the depth is $M = k + 1$.

There are no equiregular CC structures on $\mathbb{R}^2$!

⋄ Martinet distribution: $\mathbb{M} = \mathbb{R}^3$.

$H_1 = \text{span}\{X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial z} + x^2 \frac{\partial}{\partial y}\}$.

Depth $M = 3$, $x = 0$ nonregular points.

Motivation

$\diamond$ Harmonic analysis and subelliptic equations (different cases with weighted vector fields were considered in Folland, Stein 1982, Nagel, Stein, Wainger 1985, Montanari, Morbidelli 2004, Street 2011, Brandolini, Bramanti, Pedroni 2011 etc.)

$\diamond$ Optimal control theory

$$\dot{x} = f(x,u), x \in \mathbb{M}, u(t) \in \mathbb{R}^m \quad (1)$$

(conditions of controllability: Sussmann, Jurdjevich 1972, Lobry 1974, Sussmann 1977, Coron 1996 etc.)

◇ Affine control systems ($x \in \mathbb{M}^N$, $m < N$)

$$\dot{x} = \sum_{i=1}^{m} u_i(t) X_i(x) \qquad (2)$$

is locally controllable iff $Lie\{X_1, X_2, \ldots, X_m\} = T\mathbb{M}$, i.e. the "horizontal" distribution $H\mathbb{M} = \{X_1, X_2, \ldots, X_m\}$ is bracket-generating:

• span$\{X_I(v) : |I| \leq M\} = T_v\mathbb{M}$ for all $v \in \mathbb{M}$, where $X_I = [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]$, $|I| = k$ (Hörmander's condition)

• M is the depth of the sub-Riemannian space $\mathbb{M}$

• Rashevsky-Chow theorem $\Rightarrow$ on $\mathbb{M}$ there exists an intrinsic metric $d_c(u,v) = \inf_{\gamma-\text{horizontal}\,\gamma(0)=u,\gamma(1)=v} \{L(\gamma)\}$

⋄ The necessary condition of local controllability of the nonlinear system

$$\begin{cases} \dot{x} = f(x, u), \\ x(0) = x_0, \end{cases} \tag{3}$$

is that

$$\text{span}\Big\{h(0) : h \in \text{Lie}\,\frac{\partial^{|\alpha|}}{\partial u^{\alpha}} f(0, \cdot), \alpha \in \mathbb{N}^M\Big\} = T_{x_0}\mathbb{M}$$

for some $M \in \mathbb{N}$.

◇ The necessary condition of local controllability of the nonlinear system

$$\begin{cases} \dot{x} = f(x,u), \\ x(0) = x_0, \end{cases} \tag{4}$$

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for some $M \in \mathbb{N}$. Letting

$$F_\nu = \Big\{\frac{\partial^\alpha}{\partial u^\alpha} f(0,\cdot) : |\alpha| \leq \nu\Big\}$$

and

$$H_k(q) = \text{span}\{[X_1,[X_2,\ldots,[X_{i-1},X_i]\ldots](q) : X_j \in F_{\nu_j}, \nu_1+\nu_2+\ldots+\nu_i \leq k\},$$

one obtains a weighted filtration

$$H_1 \subseteq H_2 \subseteq \ldots \subseteq H_M = T\mathbb{M}, \text{ such that } [H_i,H_j] \subseteq H_{i+j}$$

more general than the Hörmander condition

Examples with nonlinear dependence on the control parameters in the system $\dot{x} = f(x, u)$ can be found e.g. in the work of Bressan, Rampazzo 2010.

In the following mechanical models the dependence is quadratic $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\dot{u}_i + \sum_{i,j=1}^{m} h_{i,j}(x)\dot{u}_i\dot{u}_j$ where $\dot{u}_i$ are the controls

$\diamond$ pendulum with oscillating pivot

$\diamond$ double pendulum with moving pivot

$\diamond$ a bead sliding without friction along a bar

Problems

1. In a neighborhood of a nonregular point, the basis $Y_1, Y_2, \dots, Y_N$, associated to the filtration $H_1 \subseteq H_2 \subseteq \dots \subseteq H_M$, varies discontinuously from point to point.

2. In the case of a weighted filtration the intrinsic Carnot-Carathéodory metric d_c might not exist.

Example (Stein "Harmonic Analysis")

$\mathbb{M} = \mathbb{R}^N$ with standard basis $\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_N}$.

Let $\deg(\partial_{x_i}) = 1$ for $1 \le i \le m$; $\deg(\partial_{x_i}) > 1$ for $i > m$.

Evidently, $H_i = \text{span}\{\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_i}\}$ satisfy $[H_i, H_j] \subseteq H_{i+j}$, since $[H_i, H_j] = \{0\}$.

But $H_1 = \text{span}\{\partial_{x_i}\}_{i=1}^m$ (for any $m < N$) does not span $\mathbb{R}^N$.

3. Different choices of weights may lead to different combinations of regular and nonregular points.

Example

$\mathbb{M} = \mathbb{R}^3$; vector fields $\{X_1 = \partial_y, X_2 = \partial_x + y\partial_t, X_3 = \partial_x\}$.

Nontrivial commutator: $[X_1, X_2] = \partial_t$.

1. Let $\deg(X_i) := 1$, $i = 1, 2, 3$. Then $\deg([X_1, X_2]) = 2$ and

$$H_1 = \text{span}\{X_1, X_2, X_3\}, \; H_2 = H_1 \cup \text{span}\{[X_1, X_2]\}.$$

In this case $\{y = 0\}$ is a plane consisting of nonregular points.

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2. Let $\deg(X_1) := a$, $\deg(X_2) := b$, $\deg(X_3) := a + b$, $a \leq b$.
Then $\deg([X_1, X_2]) = a + b \Rightarrow$

$H_a = \text{span}\{X_1\}$, $H_b = H_a \cup \text{span}\{X_2\}$, $H_{a+b} = H_a \cup H_b \cup \text{span}\{X_3, [X_1, X_2]\}$

In this case all points of $\mathbb{R}^3$ are regular.

Metric structure

We work with the following quasimetric **Nagel, Stein, Wainger 1985:**

$$\rho(v,w) = \inf\{\delta > 0 \mid \text{ there is a curve } \gamma : [0,1] \to U \text{such that}$$

$$\gamma(0) = v, \gamma(1) = w, \dot{\gamma}(t) = \sum_{|I|_h \leq M} w_I X_I(\gamma(t)), |w_I| < \delta^{|I|_h}\}.$$

Here $X_I = [X_{i_1}, [\ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]$, where $I = (i_1, \ldots, i_k)$; $|I|_h = d_{i_1} + \ldots + d_{i_k}$.

For the equiregular case $\rho(v,w) = \max_{i=1,\ldots,N} \{|v_i|^{\frac{1}{\deg Y_i}}\} = d_\infty(v,w)$

Properties:

- $d_X(u,v) \leq Q_X(d_X(u,w)+d_X(w,v))$ generalized triangle inequality

- The "Rolling-of-the-box Lemma" For all $u, v \in U$ and $r, \xi > 0$

$$\bigcup_{x \in B^\rho(v,r)} B^\rho(x,\xi) \subseteq B^\rho(v, r + C\xi + O(r^{1+\frac{1}{M}}) + O(\xi^{1+\frac{1}{M}})).$$

- $\Rightarrow d_X(u,v) \leq Q_X d_X(u,w) + d_X(w,v)$

Basic considerations

- Choice of basis $\{Y_1, Y_2, \ldots, Y_N\}$ among $\{X_I\}_{|I|_h \le M}$:

* $Y_1, Y_2, \ldots, Y_N$ are linearly independent at u (hence in some neighborhood $U(u)$);

* $\sum_{i=1}^{N} \deg Y_i$ is minimal;

* $\sum_{j=1}^{N} |I_j|$ is minimal, where $Y_j = X_{I_j}$.

- Coordinates of the second kind $\Phi^u : \mathbb{R}^N \to U$

$$\Phi^u(x_1, \ldots, x_N) = \exp(x_1 Y_1) \circ \exp(x_2 Y_2) \circ \ldots \circ \exp(x_N Y_N)(u)$$

- $\{\widehat{X}_I^u\}_{|I|_h \leq M}$ – nilpotent approximations of $\{X_I\}_{|I|_h \leq M}$ at $u \in U$.

$H_j(u) = \widehat{H}_j(u)$, where $H_j =$ span$\{X_I^u\}_{|I|_h \leq j}$, $\widehat{H}_j =$ span$\{\widehat{X}_I^u\}_{|I|_h \leq j}$.

- Quasimetic

$$\rho^u(v,w) = \inf\{\delta > 0 \mid \text{ there is a curve } \gamma : [0,1] \to U,$$

$$\gamma(0) = v, \gamma(1) = w, \dot{\gamma}(t) = \sum_{|I|_h \leq M} w_I \widehat{X}_I^u(\gamma(t)), |w_I| < \delta^{|I|_h}\}.$$

Properties:

- Conical property

$$\rho^u(\Delta^u_\varepsilon v, \Delta^u_\varepsilon w) = \varepsilon \rho^u(v, w)$$

where Δ^u_ε are dilations induced by the homogeneous weight structure.

- Rolling-of-the-box lemma $\bigcup_{x \in B^{\rho^u}(v,r)} B^{\rho^u}(x, \xi) \subseteq B^{\rho^u}(v, r + C\xi)$,

- Let $u, v \in U$, $r > 0$. Then

$$B^\rho(v, r) \subseteq B^{\rho^u}(v, r + CR(u, v, r)),$$

$$B^{\rho^u}(v, r) \subseteq B^\rho(v, r + CR(u, v, r) + O(r^{1+\frac{1}{M}}) + O(R(u, v, r)^{1+\frac{1}{M}})),$$

where $R(u, v, r)$ is the divergence of integral lines.

Divergence of integral lines

Let $u, v \in U$, $r > 0$. *Divergence of integral lines* with the center of nilpotentization u on $B(v, r)$ is

$$R(u, v, r) = \max\{ \sup_{\widehat{y} \in B^{\rho^u}(v,r)} \{\rho^u(y, \widehat{y})\}, \sup_{y \in B^{\rho}(v,r)} \{\rho(y, \widehat{y})\}\} \qquad (5)$$

Here the points y and $\widehat{y}$ are defined as follows. Let $\gamma(t)$ be an arbitrary curve such that

$$\begin{cases} \dot{\gamma}(t) = \sum\limits_{|I|_h \leq M} b_I X(\gamma(t)), \\ \gamma(0) = v, \gamma(1) = y, \end{cases}$$

and

$$\rho^u(v, \widehat{y}) \leq \max_{|I|_h \leq M} \{|b_I|^{1/|I|_h}\} \leq r.$$

$\widehat{y} = \exp(\sum\limits_{|I|_h \leq M} b_I \widehat{X}_I^u)(v)$. So sup in (5) is taken over infinite set of points $\widehat{y} \in B^{\rho^u}(v, r)$ and reals $\{b_I\}_{|I|_h \leq M}$,

Results on local geometry

Theorem (Estimate of divergence of integral lines).

Let $u, v \in U$, $\rho(u,v) = O(\varepsilon)$, $r = O(\varepsilon)$ and $B^{\rho}(v,r) \cup B^{\rho^u}(v,r) \subseteq U$. Then the following estimate on the divergence of integral lines holds: $R(u,v,r) = O(\varepsilon^{1+\frac{d_1}{\max\{M,dq\}}})$.

Can be used for constructing motion planning algorithms for the nonlinear control system (2): $\dot{x} = f(x,u)$.

Theorem (Local approximation theorem).

If $u, v, w \in U$, $\rho(u,v) = O(\varepsilon)$ and $\rho(u,w) = O(\varepsilon)$, then

$$|\rho(v,w) - \rho^u(v,w)| = O(\varepsilon^{1+\frac{d_1}{\max\{M,dq\}}}).$$

Describes local behaviour of the quasimetric

Theorem (Tangent cone theorem). The quasimetric space (U, ρ^u) is the tangent cone to the quasimetric space (U, ρ) at $u \in U$; the tangent cone is isomorphic to G/H, where G is a nilpotent group.

Here the tangent cone is understood in the sense of convergence theory for quasimetric spaces (S. 2010) which is a nontrivial generalisation of the Gromov-Hausdorff convergence theory (for any bounded quasimetric spaces $d_{GH}(X, Y) = 0$).

These results generalise classical results on local geometry for Hörmander vector fields (Mitchell 1985, Gromov 1996, Bellaiche, 1996, Jean 2001 etc.) and give new proofs for them.

Methods of proofs

• Properties of nilpotent approximations and quasimetrics for equiregular C^1 C-C spaces (Karmanova, Vodopyanov 2009; Basalaev, Vodopyanov 2013; Karmanova 2013);

• Generalization and synthesis of the classical methods of embedding a sub-Riemannian manifold into a regular one (Rotshild, Stein 1976; Hörmander, Melin 1978; Goodman 1978; Hermes 1991; Bellaiche 1996; Christ, Nagel, Stein, Wainger 1999; Jean 2001; Brandolini, Bramanti, Pedroni 2011 etc.).

• Results on algebraic and analytic properties of quasimetric spaces with dilations (S. 2010; S., Vodopyanov 2011, 2013).

THANK YOU FOR YOUR ATTENTION!