Mappings with bounded distortion on roto-translation group

Maxim Tryamkin

Moscow

April 2014

Brief history

In 1960s Reshetnyak originated the theory of mappings with bounded distortion for multidimensional Euclidean spaces.

Definition. Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geqslant 2$. A mapping $f\colon \Omega \to \mathbb{R}^n$ is called *a mapping with bounded distortion (or a quasiregular mapping)* if
1) $f \in W^1_{n,\mathrm{loc}}(\Omega)$ (a regularity condition)
2) there exists a constant $1 \leqslant K < \infty$ such that the differential $Df(x)$ satisfies the distortion condition for almost all $x \in \Omega$

$$\|Df(x)\|^n \leqslant K|\det Df(x)|$$

(geometrically it means that f transforms an infinitesimal sphere to an infinitesimal ellipsoid and the ratio of the semi-major axis to the semi-minor axis does not exceed K)

Brief history

Reshetnyak proved the following result (for Euclidean spaces):

Theorem. *Every nonconstant mapping with bounded distortion is continuous, open and discrete.*

Continuity follows from the regularity condition $f \in W^1_{n,\mathrm{loc}}(\Omega)$. This condition enables us to deduce the key relation underlying the connection between mappings with bounded distortion and nonlinear potential theory: the equality

$$df^{\#}\omega = f^{\#}d\omega$$

holds in the sense of distributions, where ω is a differential form of codegree 1.

In Euclidean spaces this equality can be proved by approximating f by smooth functions.

Brief history

The equality $df^{\#}\omega = f^{\#}d\omega$ and the distortion condition $\|Df(x)\|^n \leqslant K|\det Df(x)|$ imply the coonection with nonlinear potential theory:

Theorem. *If w is a smooth solution to the equation*

$$\operatorname{div}(|\nabla w|^{n-2}\nabla w) = 0$$

in the open set W, then the composition $v = w \circ f$ is a weak solution to the equation

$$\operatorname{div}(\mathcal{A}(|\nabla v|^{n-2}\nabla v) = 0$$

on the open set $f^{-1}(W) \cap \Omega$.

Brief history

This theorem and the fact that the equation

$$\operatorname{div}\left(|\nabla w|^{n-2}\nabla w\right) = 0$$

in $\mathbb{R}^n$, $n \geqslant 2$, has a singular smooth solution ($w \to +\infty$ as $x \to 0$)

$$w = \log\frac{1}{|x|}$$

allowed Reshetnyak to get openness and discreteness of the mapping f.

Brief history

Summary:

For proving continuity we need to check that

$$df^{\#}\omega = f^{\#}d\omega$$

For proving discreteness and openness we have additionally to exhibit a singular solution (like $\log \frac{1}{|x|}$) to the equation

$$\operatorname{div}\left(|\nabla w|^{n-2}\nabla w\right) = 0$$

Brief history

Heinonen and Holopainen proved Reshetnyak's theorem for $\mathbb{H}$-type Carnot groups but in their paper smoothness condition on f was too restrictive (1997).

Applying Reshetnyak's scheme Dairbekov established topological properties of mappings with bounded distortion for Heisenberg groups under minimal smoothness condition (2000). He tried to extend Reshetnyak's method to Carnot groups but his attempt led to unwanted side effects: an appropriate mollifier approximation gives smooth mappings which do not preserve horizontal structure.

Vodopyanov (2005) developed a new idea which allows to obtain the equality $df^{\#}\omega = f^{\#}d\omega$ without involving any approximation of f.

Roto-translation group

The *roto-translation group* $SE(2)$ is the group of motions of the Euclidean plane. It is a three-dimensional topological manifold diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$ with the group law

$$(x_0, y_0, \theta_0) \cdot (x, y, \theta) = (x_0 + x\cos\theta_0 - y\sin\theta_0, y_0 + x\sin\theta_0 + y\cos\theta_0, \theta + \theta_0)$$

and left-invariant vector fields

$$X_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial\theta}, \quad X_3 = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}$$

Commutation relations

$$[X_3, X_2] = X_1, \quad [X_1, X_3] = 0, \quad [X_2, X_1] = X_3$$

Roto-translation group

The sub-Riemannian structure on $SE(2)$ is defined by horizontal subbundle $H = \mathrm{span}\,\{X_1, X_2\}$

We introduce an inner product on H such that X_1 and X_2 are orthonormal.

The sub-Riemannian metric d_c is defined as the infimum of lengths of all horizontal curves joining two points.

By Rashevskii-Chow theorem the metric d_c is well defined because

$$\mathrm{span}\,\{X_1, X_2, [X_2, X_1] = X_3\} = T\,SE(2)$$

Sobolev mappings

First consider a function $u\colon \Omega \to \mathbb{R}$, $\Omega \subset SE(2)$. We say that $u \in W_p^1(\Omega)$, $1 \leqslant p < \infty$, if $u \in L_{1,\mathrm{loc}}(\Omega)$ and has distributional derivatives $X_j u$ along the horizontal vector fields X_j, $j = 1, 2$:

$$\int_\Omega X_j u \varphi \, d\mu = -\int_\Omega u X_j \varphi \, d\mu, \quad j = 1, 2$$

for an arbitrary test function $\varphi \in C_0^\infty$, and the finite norm

$$\|u\|_{W_p^1(\Omega)} = \left(\int_\Omega |u|^p \, d\mu \right)^{1/p} + \left(\int_\Omega |\nabla_h u|^p \, d\mu \right)^{1/p}$$

Here $\nabla_h u = (X_1 u) X_1 + (X_2 u) X_2$ is the *subgradient* of u, and $d\mu$ denotes the Haar measure on $SE(2)$ (the Lebesgue measure on $\mathbb{R}^3$)

Sobolev mappings

Consider a mapping $f\colon \Omega \to SE(2)$, $\Omega \subset SE(2)$. We say that f *belongs to the Sobolev class* $W^1_{p,\mathrm{loc}}(\Omega)$, if the following conditions hold:

1) the function $[f]_z\colon \Omega \ni x \mapsto d_c(f(x), z)$ belongs to the class $W^1_{p,\mathrm{loc}}$ for every $z \in SE(2)$;

2) the family of functions $(\nabla_h [f]_z)_{z \in SE(2)}$ has a dominant in $L_{p,\mathrm{loc}}(\Omega)$, i. e., there is a function $g \in L_{p,\mathrm{loc}}(\Omega)$ independetn of z and such that $|\nabla_h [f]_z(x)| \leqslant g(x)$ for almost all $x \in \Omega$.

Horizontal differential forms

Let dX_1, dX_2, dX_3 be 1-forms dual to the vector fields X_1, X_2, X_3, i. e.

$$dX_i(X_j) = \delta_{ij}$$

A horizontal form (of codegree 1) is a form of the following type

$$\omega(g) = a_1(g)\, dX_2 \wedge dX_3 + a_2(g)\, dX_1 \wedge dX_3$$

The pull-back

Let $f\colon \Omega \to SE(2)$ be a Sobolev mapping of class $W^1_{4,\mathrm{loc}}(\Omega)$, and let ω is a horizontal differential form. The *pull-back* $f^{\#}\omega$ is defined as follows

$$f^{\#}\omega(g)(v_1, v_2) = \omega(f(g))(Df(g)(v_1), Df(g)(v_2))$$

for $g \in \Omega$ and $v_1, v_2 \in T_g SE(2)$

Remark. If $f \in W^1_{4,\mathrm{loc}}(\Omega)$, then f is contact.

Here $Df(g)$ is the approximate differential of the mapping f at the point g, i. e.,

$$\operatorname{ap}\lim_{h\to 0} \frac{d_c(f(gh), f(g)Df(g)(h))}{d_c(0,h)} = 0$$

$df^{\#}\omega = f^{\#}d\omega$

Theorem. *Let* $f\colon \Omega \to SE(2)$, $\Omega \subset SE(2)$, *be a mapping in* $W^1_{4,\mathrm{loc}}(\Omega)$, $f(\Omega) \subset D$, *where* $D \subset SE(2)$ *is an open set, and let* $V : D \to \mathbb{R}^2$ *be a vector field* $V = (v_1, v_2) \in C^1(D)$ *such that* $\operatorname{div}_h V = X_1 v_1 + X_2 v_2$ *is bounded on* D, *and*

$$\omega(g) = v_1\, dX_2 \wedge dX_3 - v_2\, dX_1 \wedge dX_3.$$

Then the equality $df^{\#}\omega = f^{\#}d\omega$ *holds in the sense of distributions.*

An equivalent formulation

The previous theorem will be proved if we establish the following

Proposition. *The first 2 columns of the matrix* $\operatorname{ad} Df(g)$ *are divergent free vector fields, that is,*

$$\int_\Omega (A_{1k} X_1 \varphi + A_{2k} X_2 \varphi)\, d\mu(g) = 0, \quad k = 1, 2$$

for an arbitrary $\varphi \in C_0^\infty(\Omega)$

Here $\operatorname{ad} Df(g) = (A_{jk}(g)), j, k = 1, 2, 3$, is *the adjugate matrix of* $Df(g)$ defined by the relation

$$Df(g) \cdot \operatorname{ad} Df(g) = \det Df(g) \cdot \mathrm{Id}$$

Idea of proof

For proving the equality

$$\int_\Omega (A_{1k} X_1 \varphi + A_{2k} X_2 \varphi)\, d\mu(g) = 0, \quad k = 1, 2$$

we can not use approximation of f by smooth functions.

The idea (suggested by Prof. Vodopyanov) is based on using the change-of-variable formula with topological degree.

Change-of-variable formula

For diffeomorphism φ the usual change-of-variable formuls holds:

$$\int_\Omega u(\varphi(x))|J(\varphi,x)|\,dx = \int_{\varphi(\Omega)} u(y)\,dy$$

We can generalize this formula to non injective mappings:

$$\int_\Omega u(\varphi(x))J(\varphi,x)\,dx = \int_{\mathbb{R}^n} u(y)\mu(\varphi,y)\,dy$$

where

$$\mu(\varphi,y) = \sum_{x\in\varphi^{-1}(y)} \operatorname{sign} J(\varphi,x)$$

is topological degree of φ at y.

The change-of-variable formula with topological degree holds if the mapping φ satisfies the following conditions:

continuous

differentiable a.e.

possesses Lusin's property $\mathcal{N}$, i.e., the image of null measure set is also a null measure set

For $f \in W^1_{p,\mathrm{loc}}(\Omega)$ where $p > n$ we have all these properties thanks to Sobolev embedding theorems.

But we have $f \in W^1_{4,\mathrm{loc}}(\Omega)$ where $p = 4$ (not $p > 4$) and Sobolev theorems do not work.

The idea: we apply descent in the dimension.

Coarea formula

We decrease the dimension of the domain of $f\colon \Omega \to SE(2)$ using Coarea formula (Karmanova, Vodopyanov). *For all C^∞-smooth functions φ and all nonnegative measurable functions u*

$$\int_\Omega u(g)|\nabla_h \varphi(g)|\, d\mu(g) = \int_{-\infty}^{\infty} dt \int_{\varphi^{-1}(t)} u(h)\, d\mathcal{H}_c^3(h)$$

where $\varphi^{-1}(t)$ is a hypersurface of codegree 1.

Projection

We decrease the dimension of the range of $f\colon \Omega \to SE(2)$ using the notion of a projection on $SE(2)$.

Definition 1. Let $g \in SE(2)$ and

$$v = v_1X_1(g) + v_2X_2(g) + v_3X_3(g) \in T_gSE(2)$$

Let us denote

$$T_{g,k}SE(2) = \{v \in T_gSE(2) \mid v_k = 0\}, \quad k = 1, 2$$

Define the progection $Pr^t_{g,k}\colon T_gSE(2) \to T_{g,k}SE(2)$ in the tangent space by formula

$$Pr^t_{g,k}(v) = v - v_kX_k(g)$$

Projection

Definition 2. Let $g = \exp(\xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3)(0) \in SE(2)$. Denote

$$SE(2)_{a,k} = \{g \in SE(2) \mid \xi_k = \alpha_k\}, \quad k = 1, 2$$

where $a = \exp(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3)(0)$ is a fixed point. In particular,

$$SE(2)_{0,k} = \{g \in SE(2) \mid \xi_k = 0\}$$

Define the projection $Pr_k \colon SE(2) \to SE(2)_{0,k}$ in the group by the equality

$$Pr_k(g) = g \cdot \exp(-\xi_k X_k)(0)$$

Coordinates of the first kind

We can compute coordinates of the first kind on $SE(2)$

If $g = (x, y, \theta)$ then $g = \exp(\xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3)(0)$ where

$$\xi_1 = \frac{\theta}{2}\left(y + x\frac{\sin\theta}{1-\cos\theta}\right)$$

$$\xi_2 = \theta$$

$$\xi_3 = \frac{\theta}{2}\left(-x + y\frac{\sin\theta}{1-\cos\theta}\right)$$

Properties of projections

Proposition. *The mappings* $Pr_k^t\colon T_gSE(2)\to T_{g,k}SE(2)$ *and* $Pr_k\colon SE(2)\to SE(2)_{0,k}$ *possesses the following properties:*

1) the Riemannian differential $(Pr_k)_*$ *of* Pr_k *at 0 is equal to* $Pr_{0,k}^t$

2) for each fixed $a\in SE(2)$*, the Riemannian differential of* $L_a\circ Pr_k$ *at 0 equals* $Pr_{a,k}$

3) $Pr_k(a\cdot b)=Pr_k(a\cdot Pr_k(b))$ *for all* $a,b\in SE(2)$

4) for each fixed $a\in SE(2)$*,* $Pr_k\colon SE(2)_{a,k}\to SE(2)_{0,k}$ *is a sense-preserving diffeomorphism and* $\mathcal{H}^3(Pr_k(A))=\mathcal{H}^3(A)$ *for every* $\mathcal{H}^3$*-measurable set* $A\subset SE(2)_{a,k}$

5) $\mathcal{H}^3(Pr_k(A))=0$ *for every* $A\subset SE(2)$ *if* $\mathcal{H}^3(A)=0$

Change-of-variable formula on SE(2)

The previous proposition and some properties of the restriction of f to $\varphi^{-1}(t)$ allows us to establish that for the mapping $Pr_k \circ f\colon M \to SE(2)_{0,k}$, where M is a connected component of $\varphi^{-1}(t)$, the following formula holds

$$\int_M u(x) J(x, Pr_k \circ f|_M)\, d\mathcal{H}^3(x) = \int_{SE(2)_{0,k}} \sum_{x \in f^{-1}(y)} u(x)\, d\mathcal{H}^3(y)$$

$df^{\#}\omega = f^{\#}d\omega$

By the coarea formula and the change-of-variable formula the expression

$$\int_{\Omega} (A_{1k}X_1\varphi + A_{2k}X_2\varphi)\, d\mu(g)$$

can be written in the following way:

$$\int_{SE(2)_{0,k}} \mu(y, Pr_k \circ f|_M)\, d\mathcal{H}^3(y)$$

but $\mu = 0$ since the mapping $Pr_k \circ f|_M$ is homotopic to a constant mapping.

Results

the change-of-variable-formula on $SE(2)$

the equality $f^{\#} d\omega = df^{\#}\omega$

After establishing the equality $f^{\#} d\omega = df^{\#}\omega$ we can prove the continuity of the mapping f following Reshetnyak's footsteps.

Plans

What about openness and discreteness?

We need to find a superharmonic function $u\colon SE(2) \to \mathbb{R}$, i.e.

$$\mathrm{div}_h(|\nabla_h u|^2 \nabla_h u) \leqslant 0$$

And one more question.

There exist mappings with bounded distorton on $SE(2)$ which are homeomorphism: for example, left translation and reflections. Are there mappings with bounded distortion with the branch set? The branch set is $\{x \in \Omega \mid f \text{ is not locally homeomorphic at } x\}$